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**Deformations of
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Tree Singularities: Limits, Series and Stability

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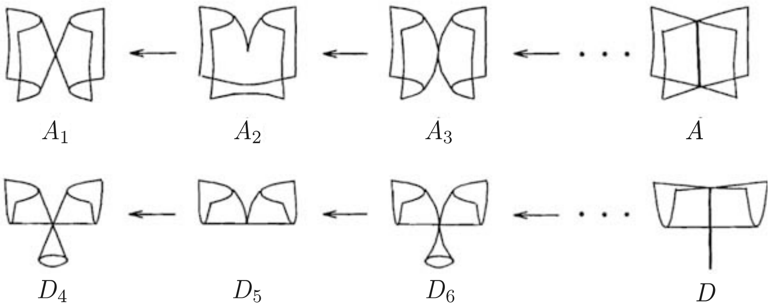
A tree singularity is a surface singularity that consists of smooth components, glued along smooth curves in the pattern of a tree. Such singularities naturally occur as degenerations of certain rational surface singularities. To be more precise, they can be considered as *limits* of certain *series* of rational surface singularities with reduced fundamental cycle. We introduce a general class of limits, construct series deformations for them and prove a stability theorem stating that under the condition of *finite dimensionality* of T^2 the base space of a semi-universal deformation for members high in the series coincides up to smooth factor with the "base space of the limit". The simplest tree singularities turn out to have already a very rich deformation theory, that is related to problems in plane geometry. From this relation, a very clear topological picture of the Milnor fibre over the different components can be obtained.

Introduction

The phenomenon of *series of isolated singularities* has attracted the attention of many authors. It is obligatory to quote Arnol'd ([3], Vol. I, p. 243):

> *"Although series undoubtedly exist, it is not altogether clear what it means."*

The very formulation is intended to be vague, and should maybe remind us that mathematics is an experimental science, and only forms concepts and definitions in the course of exploration and discovery. In any case, the word *series* is used to denote a collection of singularities $\{X_i\}_{i\in I}$, where I is some partially ordered set, which "belong together in some sense". The archetypical examples are the A_k and D_k series of surface singularities:



There are several ad hoc ways of saying that these singularities "belong together", but to quote Arnol'd again (p. 244):

"However a general definition of series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicity (for example $D \sim x^2y, T \sim xyz$), so that the hierarchy of series reflects the hierarchy of non-isolated singularities."

Most attempts have been to formalize certain aspects of the series phenomenon. D. Siersma and R. Pellikaan started studying hypersurface singularities with one-dimensional singular locus ([51], [52], [36], [38], [39]). These objects can be thought of as the *limits* of the simplest types of series of isolated singularities. A precision of this limit idea can be found in the notion of *stem,* due to D. Mond ([40]). In the thesis of R. Schrauwen [49] the notion of series is developed for plane curves from a topological point of view. It would be interesting to extend these ideas to isolated hypersurface singularities of arbitrary dimension.

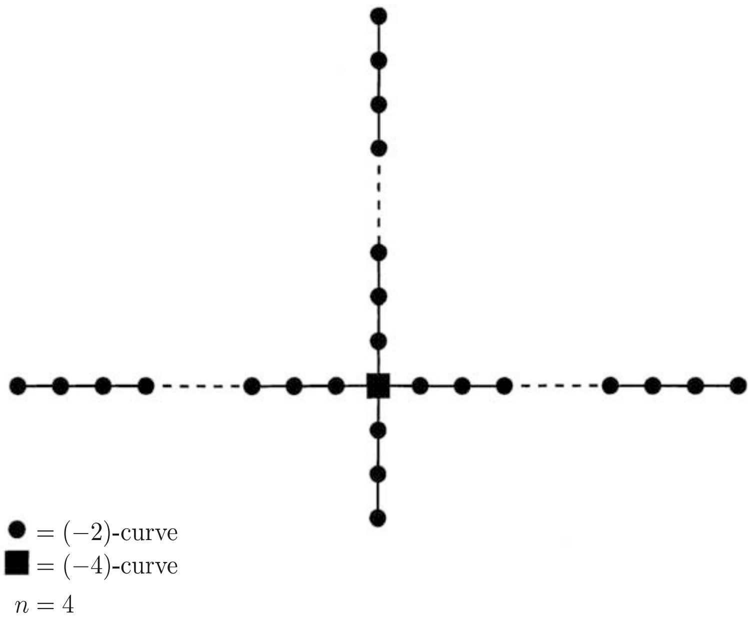
The series phenomenon was observed by Arnol'd for hypersurfaces, but for non-hypersurfaces series also undoubtedly exist; for this one just has to take a look at the tables of rational triple points obtained by M. Artin ([4]) or of the minimally elliptic singularities as compiled by H. Laufer ([30]). A series here is characterized in terms of resolution graphs: the effect of increasing the index of the series is that of the introduction of an extra (-2)-curve in a chain of the resolution. In my thesis [58] the appropriate limits for series of normal surface singularities were identified as the class of *weakly normal Cohen–Macaulay* surface singularities. So a one-index series of normal surface singularities is associated with a Cohen–Macaulay surface germ X with an irreducible curve Σ as a singular locus, transverse to which X has ordinary crossings (A_1).

Intimately related to the notions of a limit and its associated series are the ideas of *regularity* and *stability:* certain properties of the series members X_i do in fact not depend on i, at least for i big enough, and the limit X has a corresponding property. Many examples of these phenomena are known. For example, the Milnor number will grow linearly with the index [68], [39], multiplicity and geometric genus will stay constant, and the monodromy varies in a regular, predictable manner [53].

In the deformation theory of rational surface singularities one also encounters these phenomena. From the work of J. Arndt [2] on the base space of the semi-universal deformation of cyclic quotient singularities, and the work of T. de Jong and the author on rational quadruple points [23], the idea emerged of *stability of base spaces.* This is intended to mean that in a *good series* $\{X_i\}_{i\in I}$ something like the following should happen:

1. $T^1_{X_i}$ grows linearly with the index i in a series.
2. $T^2_{X_i}$ is constant (or stabilizes at a certain point).
3. "The" obstruction map $Ob : T^1_{X_i} \longrightarrow T^2_{X_i}$ becomes *independent* of the *series deformations*, and consequently
4. the base
$$\mathcal{B}_i = Ob^{-1}(0), \quad Ob : T^1_{X_i} \longrightarrow T^2_{X_i}$$
retains the same overall structure, in the sense that it gets multiplied by a smooth factor.

Of course, this is rather inprecise, but maybe the following series of singularities gives some feeling of what we are after.

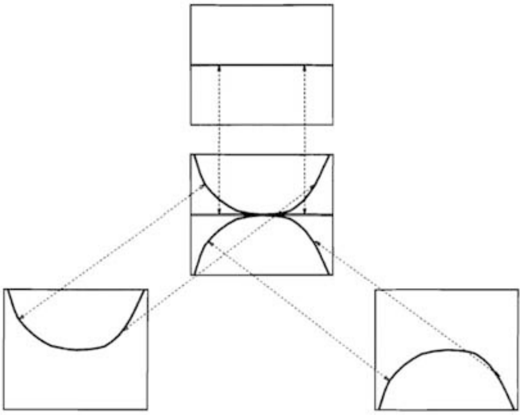


Resolution graph of a rational quadruple point

Apparently, these singularities form a four-index series $X_{a,b,c,d}$ of rational quadruple points. From [21] it follows that one has for the base space:

$$\mathcal{B}_{a,b,c,d} = B(n) \times S,$$

where $n = \min(a, b, c, d)$, $B(n)$ is a very specific space with n irreducible components and S is some smooth space germ. So, up to a smooth factor, the base space of $X_{a,b,c,d}$ only depends on the *shortest arm length n.* This n determines the *core* $B(n)$ of the deformation space, but every time we increase the shortest length, we pick up a new component! From [25], we know that T^2 is also determined by the smallest arm length. So although $X_{a,b,c,d}$ clearly "is" a four-index series, stability of the base space can only be seen by considering it as a (one-index series of a) three-index series of singularities. The *limit* obtained by sending the three shortest arms to infinity is a first example of what we call a *tree singularity:* a union of smooth planes intersecting in smooth curves in the pattern of a tree. In this case, there is a central plane with three smooth curves in it. The curves all have mutual contact of order n, and to these curves three other smooth planes are glued.



Glueing of planes

The usefulness of the stability phenomenon is obvious: as a degeneration, these limits have a simpler structure, and as a rule their deformation theory will be easier to understand than that of a series member. But one has to pay a price: as these limits no longer have isolated singularities, they do not have a semi-universal deformation in the usual sense. Their base spaces are in any sense *infinite dimensional.* This causes some inconveniences, but the work of Hauser [19] shows that a good theory can be developed in the framework of Banach-analytic spaces. We take here another approach: we will work consistently with the deformation functors and smooth transformations between them.

The purpose of this paper is twofold. In the first place it is intended as a heuristic guide to the understanding of [25]. By introducing the concept of a tree singularity we hope to clarify some of the ideas behind [25], where sometimes technicalities obscure simple and strong geometrical ideas.

In the second place we have a few theorems about series and the stability phenomenon that deserve formulation and exposition. There are many open ends here, and maybe the paper can interest others to prove more general results in this direction.

The organization of the paper is as follows. in §1 we review the basic theory of weakly normal Cohen–Macaulay surface germs. We will call such object simply *limits.* In §2 we show that such limit deforms in a *series* of normal surface singularities, whose resolution graphs can be described explicitly. Most of these ideas can be found in [58]. We will use these concepts as a sort of working definition, and in no way as the last word on these. In §3 we formulate and prove the basic stability results: the *theorem of the core* (3.5), and the *stability theorem* (3.7). The projection method of [22] is used, but clearly here is something very general going on, and a better understanding is wanted for. In §4 we take a closer look at a particular class of limits, the afore mentioned *tree singularities.* These tree singularities are the limits of series of the simplest surface singularities imaginable: those which are rational and have reduced fundamental cycle. A key notion in [25] was that of a *limit tree* of a rational singularity with reduced fundamental cycle. This is an abstraction to systematically distinguish between *long* and *short* chains of (-2)-curves in the resolution graph. Another way of thinking about a limit tree of a singularity is as an *assignment of the singularity as a series member of a limit.* Things are not always straightforward, as a singularity might very well be member of more than one series, with very different limits, unlike the situation with A_k and D_k. For these tree singularities, the deformation theory has a rather simple description. We will give an interpretation of the module generators for T^1 and T^2 as found in [25] in the case of tree singularities, and review the equations for the bases spaces. The base spaces for even the simplest tree singularities and their series members turn out to be extremely interesting, and can be interpreted in terms of elementary plane geometry. In particular, the Milnor fibre of the series members over the different components has a simple description in terms of certain configurations of curves and points. In the paper [26] with T. de Jong we have given a more systematic account of this *picture method.* With this method one now gets some insight in the dazzling complexity of deformation theory of rational singularities, and hopefully the reader will be convinced after reading this paper that the answer to the question:

"How many smoothing components does this singularity have?"

probably in most cases will be:

"Many!"

(Unless you are somewhere at the beginning of the series ...)

1. Limits and Tree Singularities

In this section we introduce a certain class of non-isolated surface singularities, called *limits.* We review some basic properties and notions of these singularities, and we will see in the next section how limits give rise to series of isolated singularities. Most of this can be found in [58]. Furthermore, we introduce a particularly simple class of limits that we call *tree singularities.*

We will consider germs X of analytic spaces, or small contractible Stein representatives thereof. Σ usually will denote the singular locus of X, and $p \in \Sigma$ the base point of the germ.

Notation 1.1. Let X be a reduced germ of an analytic space, Σ its singular locus, $\mathcal{O}_X$ its local ring, and $\mathcal{K}_X$ its total quotient ring.

The *normalization* of X is denoted by

$$n : \widetilde{X} \longrightarrow X.$$

The *weak normalization* of X is denoted by

$$w : \widehat{X} \longrightarrow X.$$

Recall that the semi-local rings of $\widetilde{X}$ and $\widehat{X}$ are given by:

$$\mathcal{O}_{\widetilde{X}} = \{f \in \mathcal{K}_X \mid f_{|X-\Sigma} \in \mathcal{O}_{X-\Sigma}\text{and } f \text{ is bounded}\}$$

$$\mathcal{O}_{\widehat{X}} = \{f \in \mathcal{O}_{\widetilde{X}} \mid f \text{ extends continuously to } X\}$$

so one has the inclusions $\mathcal{O}_X \subset \mathcal{O}_{\widehat{X}} \subset \mathcal{O}_{\widetilde{X}} \subset \mathcal{K}_X$.

A space is called *normal* if n is an isomorphism, *weakly normal* if w is an isomorphism. Normalization and weak normalization have obvious universal properties. Furthermore, the weak normalization has the property that if $h : Y \longrightarrow X$ is an holomorphic homeomorphism, then w can be factorized as $w = h \circ \overline{h}$ for some $\overline{h} : \widehat{X} \longrightarrow Y$. This explains the usefulness of the weak normalization and its alternative name *maximalization.* For more details we refer to the standard text books like [18], [14].

Example 1.2. For each m there is exactly one weakly normal curve singularity $Y(m)$ of multiplicity m, to know the union of the m coordinate lines L_p, $p = 1, \ldots, m$ in $\mathbb{C}^m$:

$$\begin{aligned} Y(m) &= \{(y_1, \ldots, y_m) \mid y_i \cdot y_j = 0,\ i \neq j\} \\ &= \cup_{p=1}^m \{(y_1, \ldots, y_m) \mid y_i = 0,\ i \neq p\} \\ &= \vee_{p=1}^m L_p \end{aligned}$$

Weakly normal surface singularities have a more complicated and interesting structure. If we assume X also to be *Cohen–Macaulay* (note that in dimension two normality implies Cohen–Macaulay, but weak normality does not), then there is a simple geometrical description of weak normality.

Definition 1.3. A Cohen–Macaulay surface germ is called a *limit* if it satisfies one of the following three equivalent conditions:

1. X is weakly normal.

2. $X - \{p\}$ is weakly normal.

3. For points $q \in X - \{p\}$ we have the following analytic local normal forms:

$$q \in X - \Sigma\,; \quad \mathcal{O}_{(X,q)} \approx \mathbb{C}\{x_1, x_2\}$$

$$q \in \Sigma - \{p\}\,; \quad \mathcal{O}_{(X,q)} \approx \mathbb{C}\{x, y_1, \ldots, y_m\}/(y_i \cdot y_j;\ i \neq j)$$

Here, of course, m can depend on the choice of q.

Proof. The equivalence of 2. and 3. is clear in view of example (1.2). Obviously 1. ⇒ 2. and 2. ⇒ 1. follows from the fact that Cohen–Macaulay implies that all holomorphic functions on $X - \{p\}$ extend to X. ■

The following gluing construction is very useful:

Proposition 1.4. *Let be given maps of analytic spaces* $\pi : \widetilde{\Sigma} \longrightarrow \Sigma$ *and* $\iota : \widetilde{\Sigma} \longrightarrow \widetilde{X}$. *If* π *is finite and* ι *is a closed embedding, then the push-out* X *in the category of analytic spaces exists, i.e. there is a diagram*

$$\begin{array}{ccc} \widetilde{\Sigma} & \stackrel{\iota}{\longrightarrow} & \widetilde{X} \\ \pi \downarrow & & \downarrow \\ \Sigma & \longrightarrow & X \end{array}$$

with the obvious universal property. The map $\widetilde{X} \longrightarrow X$ *is also finite, and the map* $\Sigma \longrightarrow X$ *is also a closed embedding. Furthermore,* $\widetilde{X} - \widetilde{\Sigma} \approx X - \Sigma$.

We say that X is obtained from $\widetilde{X}$ by *gluing* the subspace $\widetilde{\Sigma}$ to Σ. For a proof, see [27] or [58], where in the local case explicit algebra generators of $\mathcal{O}_X$ are given.

The above construction is also "universal" in the sense that any finite, generically 1-1 map $\widetilde{X} \longrightarrow X$ between reduced spaces can be obtained that way. To formulate this more precisely, we will fix the following notation associated to such a map:

Notation 1.5. Given a finite, generically 1-1 map $\widetilde{X} \longrightarrow X$ between reduced spaces, we define the *conductor* to be:

$$\mathcal{C} = \mathrm{Hom}_X(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) \subset \mathcal{O}_X.$$

Put $\mathcal{O}_\Sigma = \mathcal{O}_X/\mathcal{C}$ and $\mathcal{O}_{\widetilde{\Sigma}} = \mathcal{O}_{\widetilde{X}}/\mathcal{C}$ for the structure sheaves of the corresponding sub spaces $\widetilde{\Sigma}$ and Σ. It is now a tautology that we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & 0 \\ & & \approx\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \mathcal{O}_{\widetilde{\Sigma}} & \longrightarrow & 0 \end{array}$$

so X can be seen as obtained from $\widetilde{X}$ by gluing $\widetilde{\Sigma}$ to Σ.

This now leads to a characterization of limits in terms of the normalization:

Proposition 1.6. *Let X be a surface germ, $n : \widetilde{X} \longrightarrow X$ the normalization. Then X is a limit if and only if:*

1. $\widetilde{X}$ *is purely two-dimensional.*

2. Σ *and* $\widetilde{\Sigma}$ *are* reduced *curve germs (with structure as in 1.5).*

3. $H^0_{\{0\}}(\mathcal{O}_{\widetilde{\Sigma}}/\mathcal{O}_\Sigma) = 0$, *i.e.* $\mathcal{O}_{\widetilde{\Sigma}}/\mathcal{O}_\Sigma$ *is* $\mathcal{O}_\Sigma$*-torsion free.*

Proof. (See [58], (1.2.20)) If X is a limit, then $\widetilde{X}$ will be a normal surface (multi-) germ, and the curves Σ and $\widetilde{\Sigma}$ will be reduced, by the local normal forms 1.3 and the fact that $\mathcal{C}$ is defined as a Hom. Via a local cohomology computation using the push-out diagram, the Cohen–Macaulayness of X comes down to condition 3. ■

Partition singularities 1.7.

Suppose that we have a germ X that sits in a push-out diagram as in (1.4):

$$\begin{array}{ccc} \widetilde{\Sigma} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \hookrightarrow & X \end{array}$$

Suppose furthermore that $\widetilde{X}$, $\widetilde{\Sigma}$, and Σ are all *smooth* (multi-) germs. Hence, Σ is a single smooth branch, and $\widetilde{\Sigma}$, $\widetilde{X}$ both consist of r smooth pieces, where r is the number of irreducible components of X. The map $\widetilde{\Sigma} \hookrightarrow \widetilde{X}$ is the standard inclusion, and in appropriate coordinates the map $\widetilde{\Sigma} \longrightarrow \Sigma$ is given by $t_i \mapsto t_i^{m_1}$. Hence, X is completely described by the partition of $m = \Sigma_{i=1}^r m_i$ into r numbers. We call X a *partition singularity*, and write $X = X(\pi)$, where $\pi = (m_1, m_2, \ldots, m_r)$. This space has the following more or less obvious properties:

$$\operatorname{mult}\big(X(\pi)\big) = m, \quad \operatorname{embdim}\big(X(\pi)\big) = m+1, \quad \operatorname{type}\big(X(\pi)\big) = m-1.$$

The singular locus of $X(\pi)$ is the line Σ, and the generic transverse singularity is the curve $Y(m)$ of (1.2). The general hyperplane section of $X(\pi)$ is the partition curve of type π as defined in [11]. These partition singularities are in some sense the building blocks from which all other limits are constructed, see (2.3). Note also that $X(1,1) = A_\infty$ and $X(2) = D_\infty$.

Projections 1.8. Consider a normal surface singularity $\widetilde{X} \subset \mathbb{C}^N$, and consider a general linear projection $L : \mathbb{C}^N \longrightarrow \mathbb{C}^3$. Let X be the image of $\widetilde{X}$ in $\mathbb{C}^3$. Then X will have an ordinary double curve outside the special point. As a hypersurface X is Cohen–Macaulay, hence X is a limit, and moreover, the map $l : \widetilde{X} \longrightarrow X$ can be identified with the normalization map.

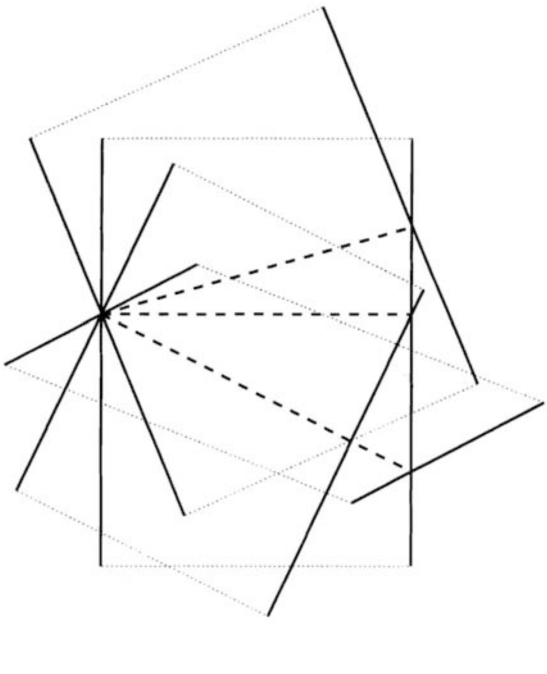
In the proof of theorem (3.5) we will use a slightly more general situation in which $\widetilde{X}$ is assumed to be a limit rather than a normal space. The corresponding X will be a limit if and only if $\widetilde{X}$ has only transverse A_1 outside the special point.

Tree singularities 1.9. A *tree singularity* is a singularity X that satisfies one of the following equivalent conditions:

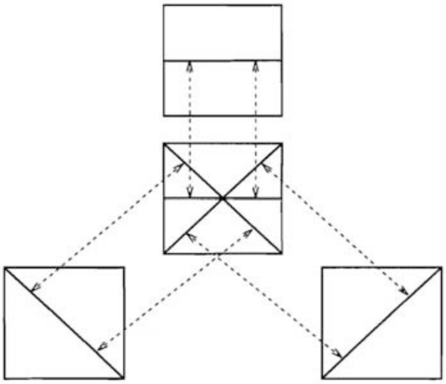
1. X is the total space of a δ-constant deformation of the curve $Y(m)$ of (1.2) to a curve with only nodes. Note that the curve $Y(m)$ has δ-invariant equal to $m-1$, and only deforms into singularities $Y(k)$, with $k \leq m$ ([12]). Note that the general fibre will have m components and $m-1$ nodes, so the components have to intersect in the pattern of a tree.

2. X has the curve $Y(m)$ of (1.2) as a general hyperplane section and is the union of m smooth irreducible components X_p, $p = 1, \ldots, m$ and $L_p \subset X_p$. Two such components X_p and X_q intersect in the point 0, or in a smooth curve $\Sigma_{\{p,q\}}$. The graph with vertices corresponding to the components X_p and edges corresponding to the curves $\Sigma_{\{p,q\}}$ is a tree T. So a tree singularity is obtained by gluing smooth planes along smooth curves in the pattern of a tree.

Example 1.10. We illustrate these two different ways of looking at a tree singularity with two pictures.



A δ-constant deformation of $Y(4)$



The same tree singularity as a glueing

To describe a tree singularity completely, we not only need T, but also a description of the curves Σ_{pq} in the planes X_p and X_q. This can be done as follows. We choose coordinates $x, y_1, y_2, \ldots, y_m$, such that the hyperplane section $x = 0$ describes $Y(m)$ in the coordinates of (1.2). As the plane X_p intersects $x = 0$ in the line L_p, the variables x, y_p form a coordinate system on X_p. As $x = 0$ is a general hyperplane section, all curves Σ_{qp} in the plane X_p are transverse to L_p, and hence are described by an equation of the form:

$$\Sigma_{qp} \; : \; y_p + a_{qp}(x) = 0$$

for some $a_{qp} \in \mathbb{C}\{x\}$. Note that the intersection multiplicity of the curves Σ_{rp} and Σ_{qp} is equal to:

$$i\left(\Sigma_{rp}, \Sigma_{qp}\right) = \mathrm{ord}_x\big(\phi(r,q;p)\big) \; =: \rho(r,q;p)$$

$$\text{where} \quad \phi(r,q;p) := a_{rp} - a_{qp}.$$

These difference functions ϕ (and the contact orders ρ) play a very important role in all sorts of computations and are to be considered as more fundamental than the a_{qp}. This leads to the following definition:

Definition 1.11. Let T be a tree and let us denote the set of vertices by $v(T)$, the set of edges by $e(T)$, and the set of oriented edges by $o(T)$. The set of corners $c(T)$ is the set of triples $(r,q;p)$ such that $\{r,p\} \in e(T)$ and $\{q,p\} \in e(T)$.

A *decorated tree* $\boldsymbol{T} = (T, \phi)$ is a tree T, together with a system ϕ of functions,

$$\phi(r,q;p) \in x\mathbb{C}\{x\} \quad (r,q;p) \in c(T)$$

anti-symmetric in the first two indices and satisfying the cocycle condition:

$$\phi(s,r;p)+\phi(r,q;p)+\phi(q,s;p)=0.$$

Furthermore, it is assumed that none of the ϕ's is identically zero.

So every tree singularity X with a function $x \in \mathcal{O}_X$ defining the general hyperplane section, gives us a decorated tree $\boldsymbol{T}$. Conversely, one has the following:

Proposition 1.12. *Let $\boldsymbol{T}=(T,\phi)$ be a decorated tree. Consider the power series ring R with variables x, z_{qp}, where $(p,q)\in o(T)$. Denote by $\boldsymbol{C}_{pq}$ the unique chain in T from p to q. Let $X(\boldsymbol{T})$ be defined by the following system of equations:*

$$\begin{aligned} z_{pq}z_{rs} &= 0 \qquad\qquad \textit{for all } (p,q),(r,s)\in o(T) \textit{ such that } p,r\in \boldsymbol{C}_{qs}\\ z_{rp}-z_{qp} &= \phi(r,q;p) \textit{ for all corners } (r,q;p)\in c(T). \end{aligned}$$

Then $X(\boldsymbol{T})$ is the tree singularity with decorated tree $\boldsymbol{T}$. The irreducible components of $X(\boldsymbol{T})$ are

$$X_t, \quad t\in v(T) \textit{ defined by } z_{sr}=0,\ s\in \boldsymbol{C}_{tr}.$$

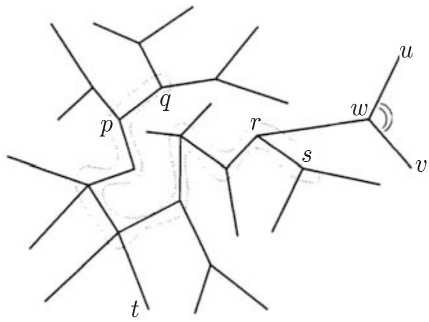
Proof. The hyperplane section of $X(\boldsymbol{T})$ is readily seen to be $Y(m)$: modulo x one has $z_{rp}=z_{qp}$, so the quadratic equations reduce to those of $Y(m)$ given in (1.2). From this it also follows that $X(\boldsymbol{T})$ is of dimension ≤ 2. Now choose a splitting of the cocycle ϕ; i.e. we write

$$z_{qp}=y_p+a_{qp}\ ;\ a_{qp}\in \boldsymbol{C}\{x\}.$$

Define X_t as the set where $z_{sr}=0$, $s\in \boldsymbol{C}_{tr}$. Then on X_t one has coordinates x, y_t and the other y_r are expressed via:

$$y_r+a_{sr}=0$$

where $s\in \boldsymbol{C}_{tr}$ is such that and $\{s,r\}\in e(T)$, so indeed X_t is a smooth surface. Furthermore, for any $t\in v(T)$ and any given (p,q) and $(r,s)\in o(T)$ such that p and $r\in \boldsymbol{C}_{qs}$ we have that $r\in \boldsymbol{C}_{ts}$ or $p\in \boldsymbol{C}_{tq}$, because T is a tree.



The tree equations $z_{pq}z_{rs} = 0$, and a corner $(u, v; w) \in c(T)$

This means that for each t the quadratic equations $z_{pq}z_{rs}$ is zero on X_t. Hence X_t is a component of $X(\boldsymbol{T})$. If $\{p, q\} \in e(T)$ then $X_p \cap X_q$ is described by the equation

$$y_p + a_{qp} = 0$$

in the plane X_p. As none of the ϕ's is identically zero, all these curves are distinct. So in each plane we find precisely the right curves to give as incidence diagram the tree T. As the hyperplane section of $X(\boldsymbol{T})$ was the reduced curve $Y(m)$, this indeed proves that $X(\boldsymbol{T})$ is the total space of a δ-constant deformation of $Y(m)$. ■

Remark 1.13. There is another, in some sense simpler, but more redundant form to write the equations for $X(\boldsymbol{T})$. We introduce for each pair $p \neq q$ a variable z_{pq} and consider the equations:

$$z_{pq}z_{qp} = 0$$

$$z_{rp} - z_{qp} = \phi(r, q; p).$$

Here we extend ϕ to all triples of distinct elements by putting: $\phi(r, q; p) := \phi(s, t; p)$ if $p \in \boldsymbol{C}_{rq}$ and where s and t are determined by the rule that $s \in \boldsymbol{C}_{rp}$ and $\{s, p\} \in e(T)$, etc. One puts $\phi(r, q; p) = 0$ in case that $p \notin \boldsymbol{C}_{qr}$. This has the effect that $z_{qp} = z_{rp}$ for such triples. This second form of the equations correspond exactly to the form used in [25], where these were called the canonical equations. The canonical equations for a rational surface singularity with reduced fundamental cycle read

$$z_{pq}z_{qp} = f_{pq}$$

$$z_{rp} - z_{qp} = \phi(r, q; p).$$

The system of functions f_{pq}, $\phi(r,q;p)$ has to satisfy a certain system of compatibility equations (the "Rim-equations"), see [25]. A fundamental fact is that the f_{pq} for $\{p,q\} \in e(T)$ and $\phi(r,q;p)$ for $(r,q;p) \in c(T)$ uniquely determine the others, so a rational surface singularity with reduced fundamental cycle is determined by data $\boldsymbol{T}, \boldsymbol{f}$, where T is a decorated tree, and $\boldsymbol{f} = \{f_{pq} \in \mathbb{C}\{x\}, \{p,q\} \in e(T)\}$ a system of functions. In this way, one can see the tree singularity as a degeneration by putting all f_{pq} for $\{p,q\} \in e(T)$ equal to zero.

2. Series of Singularities

In this section we will see how to associate with each limit X a certain (multi-) series of singularities. Such a series is constructed by deforming the singularities of an *improvement* $\pi : Y \longrightarrow X$. We will describe how the resolution graphs of the series members can be understood as *root graphs of the improvements.*

As our approach to series is based on deformation theory, it might be profitable for the reader to have a look at the appendix as well. As we want to construct series by deforming X, we first take a look at the overall structure of $\mathrm{Def}(X)$ on the infinitesimal level:

Proposition 2.1. *Let X be a limit, and Σ its singular locus and $q \in \Sigma - \{0\}$ a point of multiplicity m. If $m \neq 2$ then*

(1) $T^1_{(X,q)}$ *is a free $\mathcal{O}_{(\Sigma,q)}$-module of rank $m.(m-1)$.*

(2) $T^2_{(X,q)}$ *is a free $\mathcal{O}_{(\Sigma,q)}$-module of rank $(1/2).m.(m-1).(m-3)$.*

(If $m = 2$ these ranks are 1 and 0, respectively.) In particular, T^1_X is finite dimensional if and only if X is normal, and T^2_X is finite dimensional if and only if the multiplicity of the transverse singularities does not exceed 3.

Proof. By the local normal form (1.3), this is really a statement about the curve $Y(m)$ of (1.2). For this the calculation of T^1 and T^2 is an easy exercise. See also [17], and [11]. For more information about the semi-universal deformation of this curve, see [57]. ∎

Formally, the base space $\mathcal{B}_X$ of a limit X is the fibre over 0 of a map $Ob : T^1_X \longrightarrow T^2_X$. If T^2 is finite dimensional, then there always will be deformations, because T^1_X has infinite dimension. To get some control over these deformations it is useful to study deformations of an improvement Y of X. A normal surface singularity X can be studied effectively using a resolution, i.e. a proper map $\pi : Y \longrightarrow X$ where Y is smooth, and $\pi_*(\mathcal{O}_Y) = \mathcal{O}_X$. If X is a limit and has a curve Σ as singular locus, then one can first normalize X to get $\widetilde{X}$, and then resolve $\widetilde{X}$ to $\widetilde{Y}$. The resulting map $\pi : \widetilde{Y} \longrightarrow X$ is still proper, but because we removed the singular curve, we no longer have $\pi_*(\mathcal{O}_{\widetilde{Y}}) = \mathcal{O}_X$. In order to preserve this property, we have to "glue back" the identification of points that was lost during normalization. The prize one has to pay is that the resulting space Y now has become singular. These singularities however can be controlled. Improvements were first considered by N. Shepherd-Barron [50]. For improvements of surfaces with more general transverse singular loci, see [54].

Definition 2.2. Let X be a limit. $\pi : Y \longrightarrow X$ is called an *improvement* of X if it satisfies the following properties:

(1) π is proper.

(2) $\pi : Y - E \approx X - \{p\}$, where $E = \pi^{-1}(p)$, the exceptional locus.

(3) Y has only partition singularities.

Proposition 2.3. *Improvements exist.*

Proof. Let $n : \widetilde{X} \longrightarrow X$ be the normalization, $\Sigma \in X$ and $\widetilde{\Sigma} \in \widetilde{X}$ the locus of the conductor in X and $\widetilde{X}$ respectively. Now make an embedded resolution of $\widetilde{\Sigma}$ in $\widetilde{X}$ to get a diagram

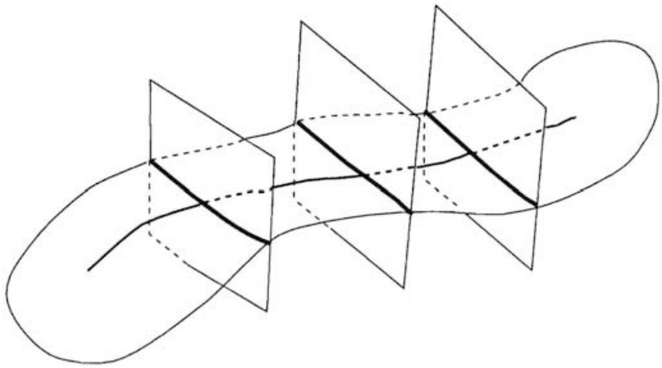
$$\begin{array}{ccc} \widetilde{\Delta} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ \widetilde{\Sigma} & \longrightarrow & X \end{array}$$

where $\widetilde{\Delta} \longrightarrow \widetilde{\Sigma}$ can be identified with the normalization of $\widetilde{\Sigma}$. By the universal property of normalization, the composed map $\widetilde{\Delta} \longrightarrow \Sigma$ now lifts to a map $\widetilde{\Delta} \longrightarrow \Delta$, where $\Delta \longrightarrow \Sigma$ is the normalization of Σ. Now we can form a push-out diagram as in 1.4:

$$\begin{array}{ccc} \widetilde{\Delta} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & Y \end{array}$$

Clearly, from the definition (1.7) we see that Y has only partition singularities. By the universal property of the push-out, we get an induced map $Y \longrightarrow X$. It is easily checked that this indeed is an improvement. ■

Example 2.4.



Improvement of the tree singularity of (1.10)

With this notion of improvement one can now try to build up a theory of limits along the same line as that exists for normal surface singularities. So one can define a fundamental cycle, weakly rational singularities, weakly elliptic singularities that all have properties very closely resembling those in the case of normal singularities ([58]). Let us recall here the definition of the geometric genus p_g:

Definition 2.5. Let X be a limit, and $\pi : Y \longrightarrow X$ an improvement. The geometric genus p_g of X is defined to be

$$p_g(X) = \dim\left(R^1\pi_*(\mathcal{O}_Y)\right).$$

It is shown in [58], (2.5.28) that this p_g is semi-continuous under deformation. To compute it in examples, the following simple result is useful:

Proposition 2.6. *If $\widetilde{X} \longrightarrow X$ is a finite, generically* 1-1 *mapping of limits, and $\widetilde{\Sigma}$, Σ as in* 1.5, *then one has*

$$p_g(X) = p_g(\widetilde{X}) + \delta_{\widetilde{\Sigma}} - \delta_{\Sigma}.$$

(Application of this to the normalization $n : \widetilde{X} \longrightarrow X$ shows that indeed p_g is independent of the chosen improvement.)

Proof. This is straightforward, see [58], (2.3.5). ■

Definition 2.7. A limit X is called weakly rational (also called semi-rational), if and only if $p_g = 0$.

Corollary 2.8. *Tree singularities are weakly rational.*

Proof. Apply (2.6) inductively to the tree singularities obtained from the given one by deleting one of its planes. ■

There are many other arguments for this fact, see [58], (4.4.6), [25], (1.4).

Definition 2.9. Let X be a limit, $\pi : Y \longrightarrow X$ an improvement and $E = \pi^{-1}(p)$ its exceptional locus. We can write $E = \cup_{i=1}^{r} E_i$ with E_i irreducible curves. By a *cycle* F on Y we mean any formal integral linear combination of the E_i. We write

$$F = \sum_{i=1}^{r} n_i E_i$$

Such a cycle determines a unique Weil-divisor (i.e. an in general non-reduced subscheme of codimension one) of Y, that we will denote by the same symbol F. The cycle F is called a Cartier-cycle, if the corresponding divisor in fact is a Cartier divisor on Y.

Definition 2.10. Let X be a limit, and $\pi : Y \longrightarrow X$ an improvement with exceptional set E and let $F \hookrightarrow Y$ the subscheme determined by a cycle on Y. Associated to this we consider the following functors.

(1) $\mathrm{Def}\,(Y)$, the deformations of Y.

(2) $\mathrm{Imp}\,(Y)$, the deformations of Y that blow down to deformations of X. This is analogous to the functor Res of [6], or B, of [65].

(3) $\mathrm{Def}\,(F \setminus Y)$, the deformations of Y for which F can be lifted as a trivial family.

This last functor is analogous to functors TR_Z considered in [65]. On Y we have a finite set P of special points, $P = \Delta \cap E$, where Δ is the singular locus of Y. We also define local functors:

(1) $\mathrm{Def}\,(Y)_P = \Pi_{p\in P}\,\mathrm{Def}\,\big((Y,p)\big)$

(2) $\mathrm{Def}\,(F\setminus Y)_P = \Pi_{p\in P}\,\mathrm{Def}\,\big((F,p)\setminus(Y,p)\big)$.

All these functors are connected and semi-homogeneous. In the case that Y is smooth, these have a hull. In general, none of these will be smooth.

Proposition 2.11.

(1) *The localization maps*

$$\mathrm{Def}\,(Y) \longrightarrow \mathrm{Def}\,(Y)_P \quad \textit{and} \quad \mathrm{Def}\,(F\setminus Y) \longrightarrow \mathrm{Def}\,(F\setminus Y)_P$$

are smooth.

(2) $\mathrm{Def}\,(Y)_P$ *is smooth if and only if* X *has finite dimensional* T^2.

(3) *There are inclusions* $\mathrm{Imp}\,(Y) \subset \mathrm{Def}\,(Y)$ *and* $\mathrm{Def}\,(F\setminus Y) \subset \mathrm{Def}\,(Y)$.

If F *is "big enough", one has* $\mathrm{Def}\,(F\setminus Y) \subset \mathrm{Imp}\,(Y)$.

Proof. Statement (1) means in particular that local deformations can be globalized, even if we want to lift the cycle F. From the local-to-global spectral sequence and using the fact that $H^2(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on Y one gets:

$$H^0(\Theta_Y) \approx \boldsymbol{T}^0(Y)$$
$$0 \longrightarrow H^1(\Theta_Y) \longrightarrow \boldsymbol{T}^1(Y) \longrightarrow H^0(\mathcal{T}^1_Y) \longrightarrow 0$$
$$\boldsymbol{T}^2(Y) \approx H^0(\mathcal{T}^2_Y).$$

From this it follows that the map $\mathrm{Def}\,(Y) \longrightarrow \mathrm{Def}\,(Y)_P$ is surjective on the level of tangent spaces, and injective (even isomorphism) on obstruction spaces. Hence the transformation is smooth. In exactly the same way one gets for $F\setminus Y$:

$$H^0\big(\Theta_Y(-F)\big) \approx \boldsymbol{T}^0(F\setminus Y)$$
$$0 \longrightarrow H^1\big(\Theta_Y(-F)\big) \longrightarrow \boldsymbol{T}^1(F\setminus Y) \longrightarrow H^0(\mathcal{T}^1_{F\setminus Y}) \longrightarrow O$$
$$\boldsymbol{T}^2(F\setminus Y) \approx H^0(\mathcal{T}^2_{F\setminus Y})$$

and the same conclusion.

Statement (2) follows directly from (2.1): T^2_X finite dimensional means that X has transverse $Y(2)$ or $Y(3)$. Hence on the improvement we find the partitiion singularities $X(1,1)$, $X(2)$, $X(1,1,1)$, $X(2,1)$ or $X(3)$. The first two are hypersurfaces, the other three Cohen–Macaulay of codimension two. So in all these cases we have $\mathcal{T}^2_Y = 0$, hence Def (Y) is smooth. As to statement (3) we remark that the inclusion Def $(F \setminus Y) \subset$ Def (Y) is due to the fact that the embedding of F in Y is unique, even infinitesimally, as a consequence of the negativity of E. That for big F, we get in fact Def $(F \setminus Y) \subset$ Imp (Y) follows from the fact that a deformation $\mathcal{Y} \longrightarrow S$ of Y over S blows down to a deformation of X if and only if $H^1(\mathcal{O}_{Y_s})$ is constant for all $s \in S$ (see [42], [62]). We say that F is *big enough* if the canonical surjection $H^1(\mathcal{O}_Y) \longrightarrow H^1(\mathcal{O}_F)$ is an isomorphism. (As $p_g = H^1(\mathcal{O}_Y)$ is finite dimensional it follows that there are such F.) ■

Remark 2.12. The most important case one encounters is of course the case that the limit has only transverse double points, so we have only $X(1,1) = \boldsymbol{A}_\infty$ and $X(2) = \boldsymbol{D}_\infty$ singularities on the improvement and so Def (Y) and Def $(F \setminus Y)$ are smooth. If furthermore the singularity is weakly rational, as is the case for our tree singularities, then one also has Def $(Y) =$ Imp (Y).

In [58] a series of determinal deformations of the partition singularities $X(\pi)$ was constructed. To be more precise, we have:

Proposition 2.13. *Let $X(\pi)$, $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ be a partition singularity. Let $(\nu_1, \nu_2, \ldots, \nu_r)$, $\nu_i \geq 0$, a collection of r numbers. Then there exists a deformation*

$$\phi : \mathcal{X}(\pi;\nu) \longrightarrow \Lambda_\pi := \mathbb{C}^r$$

such that for generic $\lambda \in \Lambda_\pi$ the fibre $X_\lambda(\pi,\nu) := \phi^{-1}(\lambda)$ has the following properties:

(1) *$X_\lambda(\pi,\nu)$ has an isolated singularity at the origin.*

(2) *The resolution graph of the minimal resolution of $X_\lambda(\pi,\nu)$ has the following structure:*

   (a) *All the curves are isomorphic to $\mathbb{P}^1$.*

   (b) *There is a central curve C, with $(C.C) = -m$.*

(c) *There are r chains of curves*

$$C_{i,1}, C_{1,2}, \ldots, C_{i,p_i}$$

where $i = 1, 2, \ldots, r$ and $p_i = \pi_i + \nu_i - 1$.

(d) $(C.C_{i,\pi_i}) = 1$.

(3) *The subscheme F_n, defined by $x^n = 0$ lifts trivially over Λ_π if*

$$n \leq \sum_{i=1}^{r} (\nu_i + 1).$$

Proof. This is essentially [58], (1.3.12). It is obtained by perturbing a matrix defining $X(\pi)$ in a very specific way. From this representation it is possible to read off all the information. ■

Remark 2.14. In the case that one or more of the ν_i is equal to 0, the fibre $X_\lambda(\pi, \nu)$ has in general more singularities. For example, $X_\lambda(\pi, 0)$ has as singularities the $(-m)$-singularity, together with $\boldsymbol{A}_{\pi_i - 1}$-singularities. This all fits with the description under (2.13) 2). Note also the special classes

$$X(1,1; a, b) = \boldsymbol{A}_{a+b+1} \quad \text{and} \quad X(2; a) = \boldsymbol{D}_{a+2}.$$

So indeed these series associated to the partition singularities are a generalization of the $\boldsymbol{A}$ and $\boldsymbol{D}$ series. But from the construction as a partition singularity we unfortunately get $\boldsymbol{A}$ as a two-index series. We will strictly hold to the following equations for the $\boldsymbol{A}$ and the $\boldsymbol{D}$ series: $\boldsymbol{A}_k : yz - x^{k+1}$ and $\boldsymbol{D}_k : z^2 - x.(y^2 - x^{k-2})$, and so $\boldsymbol{D}_3 \approx \boldsymbol{A}_3$ and the deformation of $\boldsymbol{A}_\infty$ to $\boldsymbol{A}_{-1}$ represents the generator of the T^1, etc.

Definition 2.15. Let X be a limit, and $\pi : Y \longrightarrow X$ an improvement and F a sufficiently big divisor.

Roots: In [65] the notion of *root* for a normal surface singularity was introduced. It is an attempt to characterize those cycles on a resolution that can arise as specialisation of a connected smooth curve. Analogueously we call a Cartier cycle $R \subset Y$ a root iff $\chi(\mathcal{O}_R) \leq 1$. Essentially by [65], lemma (1.2), the set of roots is always finite. A root is called *indecomposable* if it is not the sum of two other roots. In particular, each exceptional curve, not passing through any of the special points $P = E \cap \Delta$, is an indecomposable root. The diagram with vertices the indecomposable roots, and with edges corresponding to intersections of roots (computed as cycles on the normalization) we call the root diagram $RD(Y)$. Note that if Y is a resolution, then $RD(Y)$ is nothing but the dual resolution graph.

Modifications: Let $s \in P$ be a special point. Then $(Y, s) \approx \big(X(\pi), 0\big)$ for some $\pi = \pi(s) = \big(\pi_1(s), \ldots, \pi_{r(s)}(s)\big)$. In particular, the normalization of Y at s consists of $r(s)$ smooth planes. For each $s \in P$ and $i = 1, 2, \ldots, r(s)$ there are elementary modifications

$$Y_{\varepsilon_i(s)} \longrightarrow Y$$

by blowing up in the i-th piece of the normalization of (Y, s). Note that on $Y_{\varepsilon_i(s)}$ there is a unique point $\widetilde{s}$ over s at which $(Y_{\varepsilon_i(s)}, \widetilde{s}) \approx \big(X\big(\pi(s)\big), 0\big)$. In order to simplify notations we will identify the sets of special points on Y and $Y_{\varepsilon_i(s)}$. In this way we can iterate or compose these elementary transformations. The semi-group spanned by them we denote by

$$\mathcal{N}(Y) = \bigoplus_{s \in P} \bigoplus_{i=1}^{r(s)} \mathbb{N}.\varepsilon_i(s).$$

If $\nu = \big(\nu(s)\big)_{s \in P} \in \mathcal{N}(Y)$, then we denote the space obtained by this composition of elementary transformations by $Y_\nu \longrightarrow Y$. Finally, for a Cartier cycle F on Y we put

$$\mathcal{N}(Y, F) = \left\{ \nu \in \mathcal{N}(Y) \,\middle|\, \sum_{i=1}^{r(s)} (\nu_i(s) + 1) \geq \text{coeff}\,(F, s) \right\}.$$

(Here coeff (F, s) is the coefficient of E_i in F for any E_i that contains s.)

A particular transformation is the blow-up $b : \widetilde{Y} \longrightarrow Y$ of Y at s. One can check that on $\widetilde{Y}$ there is again a partition singularity of type $\pi(s)$ at some point $\widetilde{s}$ lying over s, together with $r(s)$ singularities, of type $\boldsymbol{A}_{\pi_i(s)-1}$. Let $\mu : Y_\mu \longrightarrow Y$ be the space obtained from Y by first blowing up at all the special points, and then resolve the resulting $\boldsymbol{A}$-singularities. The use of this blow-up is that on Y_μ there will be an unique indecomposable root $R(\widetilde{s})$ passing through $\widetilde{s}$. For details we refer to [58].

Series Deformations: For each $\nu \in \mathcal{N}(Y, F)$ we also get an element $\xi(\nu) \in \text{Def}\ (Y)_P(\Lambda)$ by putting together the local deformations of (2.13)

$$\xi_s\big(\nu(s)\big) \, : \, \mathcal{X}\big(\pi(s), \nu(s)\big) \longrightarrow \Lambda_{\pi(s)}$$

where $\nu = \big(\nu(s)\big)_{s\in P}$ and $\Lambda = \Pi_{s\in P}\Lambda_{\pi(s)}$.

All this was set up in such a way that the following theorem is true:

Theorem on Series 2.16. *Let $Y \longrightarrow X$ be an improvement of a limit, and F a sufficiently big divisor with* $\operatorname{supp} F = E$. *For each $\nu \in \mathcal{N}(Y, F)$ there is a deformation*

$$\mathcal{X}(\nu) \longrightarrow \Lambda$$

of X, such that for generic $\lambda \in \Lambda$ the fibre $X_\lambda(\nu)$ has an isolated singularity with resolution graph

$$\Gamma\big(X_\lambda(\nu)\big) = RD\big((Y_\lambda)_\nu\big).$$

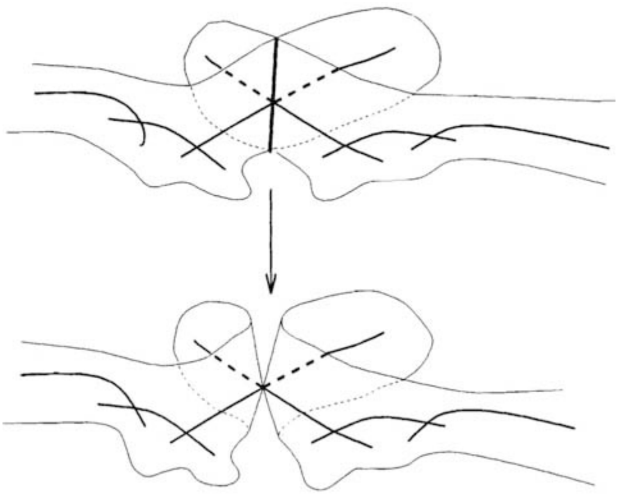
We call the singularities $X_\lambda(\nu)$ the members of the series.

Proof. The local deformations $\xi(\nu)$ can be lifted to global deformations of Y, fixing F, by (2.11). Because F is big enough, this deformation can be blown down to give a deformation of X. At first, this is only a formal deformation, over the formal completion of Λ. But by an application of the Approximation Theorem (see appendix) we can get an honest deformation over a neighborhood of zero in Λ that approximates arbitrarily good the given formal one. Because the support of F is assumed to be the full exceptional divisor, all the irreducible components E_i lift. Furthermore, locally around each point $s \in P$ we have a standard situation, producing a resolution graph as in (2.13). It is an exercise to verify that the new roots on Y_μ are the curves of the $\boldsymbol{A}$-singularities, together with the indecomposable root $R(s)$ mentioned in (2.15). This root lifts to the central curve of the local resolution. ∎

Remark 2.17. This is a rather weak theorem. We do not claim that any singularity can be degenerated to a limit, nor do we claim that all singularities with a given graph do occur as fibre $X_\lambda(\nu)$. Although this is rather plausible, it is much harder to prove. Our statement is really not much more than a statement about graphs, stated in a slightly fancy way. Note also that the construction depends on the improvement $Y \longrightarrow X$ in the following way: if we blow up further to Y_ν, $\nu \in \mathcal{N}(Y)$, then one gets as series members the $X_\lambda(\nu + \mu)$, $\mu \in \mathcal{N}(Y, F)$.

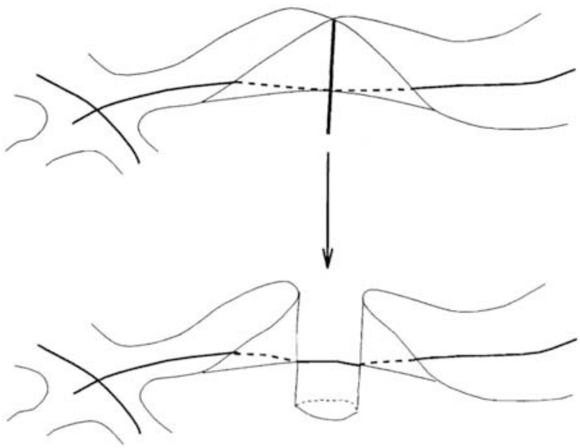
Examples 2.18. By far the most important cases are where we have only $\boldsymbol{A}_\infty$ and $\boldsymbol{D}_\infty$ singularities on the improvement. We illustrate the theorem with two pictures, that hopefully will clearify everything.

$\boldsymbol{A}$-case:



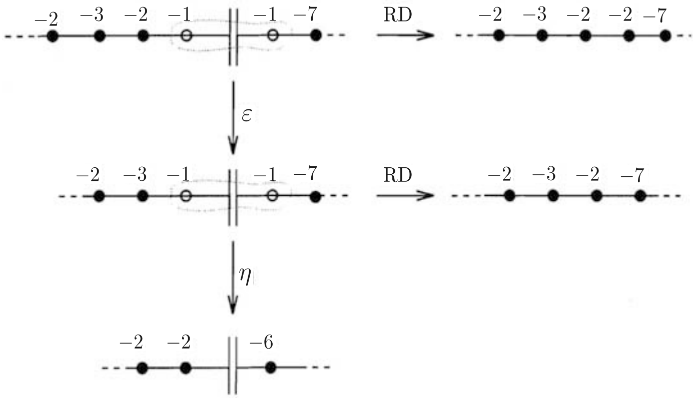
Deformation of $\boldsymbol{A}_\infty$ to $\boldsymbol{A}_1$ on improvement

$\boldsymbol{D}$-case:

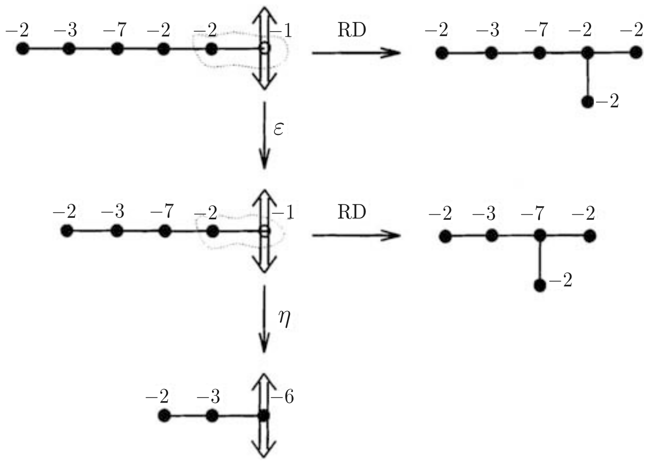


Deformation of $\boldsymbol{D}_\infty$ to $\boldsymbol{D}_2$ on improvement

We will draw improvement graphs using an obvious extension of the usual rules for drawing a resolution graph: the presence of an $\boldsymbol{A}_\infty$-singularity is indicated by a double bar. A $\boldsymbol{D}_\infty$ is indicated by a double barred arrow. We give two examples to illustrate (2.16):



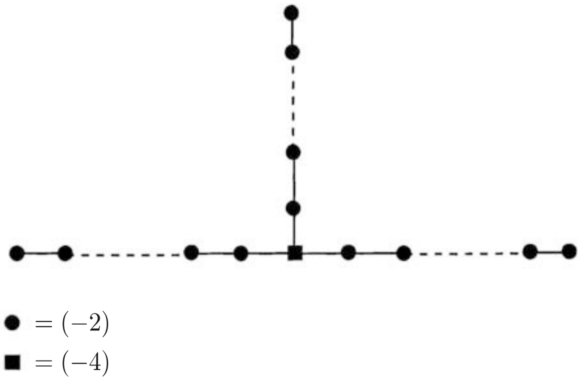
Series formation on the $\boldsymbol{A}$-case



Series formation on the $\boldsymbol{D}$-case

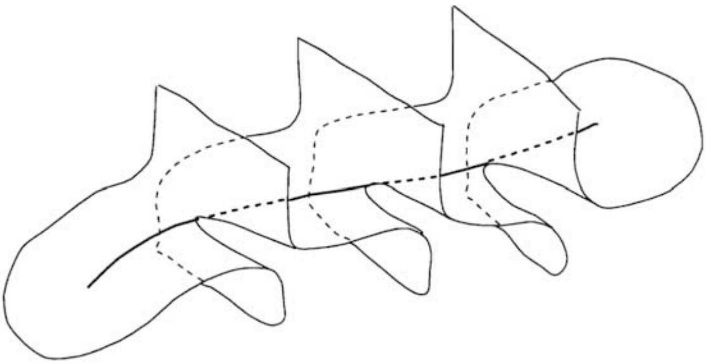
The vertical maps between the graphs are elementary modifications, obtained by blowing up the special point of the improvement. The curves enclosed by the dotted line make up the unique indecomposable root that contains the special point. The root diagrams at the right hand side are the resolution graphs of the series deformation. Blowing up further brings us to higher members of the series.

Example 2.19. The series members belonging to the improvement of (2.4) have as resolution graphs:



Series coming out of the improvement (2.4)

The construction of (2.16) lets the series begin with arm lengths equal to one, but clearly we also could let the series start one step earlier, by deforming on the improvement to $\boldsymbol{A}_0$. It looks as follows:



Deforming to $\boldsymbol{A}_0$: the (-4)

We will not make fuzz about the beginning of a series. Does the $\boldsymbol{A}_k$-series start with $k = 1$, $k = 0$, or $k = -1$?

In order to link up these series deformations with the *equations* of X we consider one more functor.

Definition 2.20. Let $\mathcal{X} \longrightarrow S$ be a deformation of X over S, with a section $\sigma : S \longrightarrow \mathcal{X}$. Let $\boldsymbol{m}_\sigma$ be the ideal of $\sigma(S) \subset \mathcal{X}$. The subscheme $J^n_\sigma(\mathcal{X})$ defined by the ideal $\boldsymbol{m}^n_\sigma$ we call the "n-jet of $\mathcal{X}$ along the section σ". We let $\mathrm{Sec}^n\,(X)(S) :=$ deformations of X over S with section, such that the n-jet of X along the section is deformed trivially over S, modulo isomorphism.

There is no difficulty in showing that this functor is connected and semi-homogeneous (see Appendix).

Proposition 2.21. *Let* $\rho : Y \longrightarrow X$ *be an improvement of a limit* X. *For all* n *there exists an* F *big enough such that the blow-down map* $\mathrm{Def}\,(F \setminus Y) \longrightarrow \mathrm{Def}\,(X)$ *factors over* $\mathrm{Sec}^n(X) \longrightarrow \mathrm{Def}\,(X)$.

Proof. We assume that the pull-back of the maximal ideal $\rho^*(\boldsymbol{m}_X)$ is invertible, say $= \mathcal{O}_Y(-Z)$. Now take $m \geq n$ such that $\boldsymbol{m}_X^m$ annihilates the sky-scraper sheaf $R^1\rho_*(\mathcal{O}_X)$, and put $F = m.Z$. This F does the job. ■

3. Stability

Before formulating the stability theorem for series deformations, let us quickly discuss a situation in which the tangent-cohomological aspect of the stability phenomenon can be readily understood.

Consider a limit X, and choose a *slicing* for it. By slicing, we mean that we consider X together with a non-constant map $\rho : X \longrightarrow S$, where S is a germ of a smooth curve. So X is sliced into curves $Y_s = \rho^{-1}(s)$, if appropriate representatives for X and S are chosen. We let $Y = \rho^{-1}(0)$. Now consider a one-parameter smoothing of X:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \pi\downarrow \\ \{0\} & \longrightarrow & T \end{array}$$

By lifting the function ρ to $\mathcal{X}$ we get a combined two-parameter deformation $\phi : \mathcal{X} \longrightarrow S \times T$ of the curve singularity Y.

Lemma 3.1. *If* $\dim(T^2_X) < \infty$, *then* $T^k_{\mathcal{X}/S\times T}$ $(k = 2, 3)$ *are artinian* $\mathcal{O}_{S\times T}$-*modules.*

Proof. We will use general properties of the cotangent complex, for which we refer to [20] and in particular to [11] for a nice summary of the most important facts. In general, the support of the $T^k_{\mathcal{X}/S\times T}$ as $\mathcal{O}_{S\times T}$-modules is contained in the discriminant D of the map $\mathcal{X} \longrightarrow S \times T$, which in our case consists of the axis $S \times \{0\}$, possibly together with some other curve C. For $p \in C - \{0\}$ the fibre $Y_{(s,t)}$ lies as a hypersurface in the smooth surface $X_t = \pi^{-1}(t)$, so we see that for $k \geq 2$ the module $T^k_{\mathcal{X}/S\times T}$ is supported on $S \times \{0\}$. As the limit X is assumed to have a finite T^2, it follows that X has only double or triple points transverse to Σ, and from this it follows that $T^2_{\mathcal{X}/S\times T}$ is concentrated over $\{0\}$, hence is artinian. But also $\operatorname{supp}(T^3_{\mathcal{X}/S\times T}) = \{0\}$, because transverse to the S axis in $S \times T$ we have a smoothing of the curve singularity $Y(3)$. As the support of T^3 of the family is concentrated over the S-axis, and vanish at the general point because $T^2_{Y(3)}$ vanishes: the usual argument. ■

Now consider the maps:

$$\iota_n : S \longrightarrow S \times T; \quad s \longmapsto \left(s, \lambda.s^{n+1}\right), \quad \lambda \neq 0$$

and let the image $\operatorname{Im}(\iota_n)$ be defined by $t_n \in \mathcal{O}_{S\times T}$. We can pull-back the family $\phi : \mathcal{X} \longrightarrow S \times T$ over the maps ι_n to get sliced surfaces $X_n \longrightarrow S$. These $X_n \longrightarrow S$ can be seen as slicings of a series of isolated surface singularities. If we let run λ over a smooth curve germ Λ, we obtain a one parameter deformations $\mathcal{X}_n \longrightarrow \Lambda$ of X with fibres the X_n. We have that $\mathcal{X}_n \longrightarrow \Lambda \in \operatorname{Sec}^n(X)(\Lambda)$, because the equations of X and X_n are the same up to order n as they are obtained by pulling back via maps that are the same up to order n.

We now can see that the obstruction spaces of the X_n stabilize in the following sense:

Proposition 3.2.

$$\lim_{n\to\infty} \dim\left(T^2_{X_n}\right) = \dim\left(T^2_X\right).$$

Proof. There exists a long exact sequence

$$\ldots \longrightarrow T^1_{\mathcal{X}/S\times T} \xrightarrow{t_n\cdot} T^1_{\mathcal{X}/S\times T} \longrightarrow T^1_{X_n/S} \longrightarrow T^2_{\mathcal{X}/S\times T}$$
$$\xrightarrow{t_n\cdot} T^2_{\mathcal{X}/S\times T} \longrightarrow T^2_{X_n/S} \cdots .$$

Here $t_n \in \mathcal{O}_{S\times T}$ is as before. We have seen that $T^k_{\mathcal{X}/S\times T}$ for $k = 2, 3$ are artinian $\mathcal{O}_{S\times T}$-modules. As a consequence, we see that $\operatorname{Ker}(t_n.)$ and

Coker $(t_n.)$, where $t_n. : T^k_{\mathcal{X}/S\times T} \longrightarrow T^k_{\mathcal{X}/S\times T}$ stabilize for $n \gg 0$. By comparing with the exact sequence

$$\cdots \longrightarrow T^1_{\mathcal{X}/S\times T} \xrightarrow{t\cdot} T^1_{\mathcal{X}/S\times T} \longrightarrow T^1_{X/S} \longrightarrow T^2_{\mathcal{X}/S\times T}$$
$$\xrightarrow{t\cdot} T^2_{\mathcal{X}/S\times T} \longrightarrow T^2_{X/S} \cdots$$

we can conclude

$$\lim_{n\to\infty} \dim\left(T^2_{X_n/S}\right) = \dim\left(T^2_{X/S}\right).$$

Because S is smooth one has $T^k_{X/S} = T^k_X$ for $k \geq 2$. (c. f. [11], (1.3.1).)

(In the case that X has only transverse double points, the same argument shows that in fact all the $T^k_{X_n}$ for $k \geq 2$ will stabilize.) ■

Remark 3.3. The above arguments show that "there are series such that high in the series T^2 (and even T^k) stabilizes, if the T^2 of the limit is finite". This is much weaker than the statement that this will happen for all deformations in $\mathrm{Sec}^n(X)$, for $n \gg 0$. Although this sounds rather probable, I have been unable to establish this. As it would be quite useful in practice, it seems worth trying to prove this in general.

As T^2_X stabilizes, is seems natural to expect that the equations of the base space also will stabilize. These stable equations then would be the equations for the base space of the limit, and these equations should not depend on the coordinates corresponding to the series deformations. So, although infinite dimensional, the base space of a limit should have some sort of finite dimensional *core.* This in fact we can prove, if we define the core in the following way:

Definition 3.4. Two semi-homogeneous functors F and G are called "the same up to a smooth factor" if there exists a semi-homogeneous functor H and smooth natural transformation $H \longrightarrow F$ and $H \longrightarrow G$. Being the same up to a smooth factor clearly is an equivalence relation.

A semi-homogeneous functor F is said to have a core iff it is the same up to a smooth factor as a (pro)-representable one. The (equivalence class) of this pro-representable functor we call the core $\mathrm{Core}\,(F)$ of F.

Theorem of the Core 3.5. *Let X be a limit with* $\dim\left(T^2_X\right) < \infty$. *then* $\mathrm{Def}\,(X)$ *has a core.*

Proof. We choose an embedding of X into some $\mathbb{C}^N$ and a linear projection $L : \mathbb{C}^N \longrightarrow \mathbb{C}^3$. Denote $L_{|X}$ by ν and put $Y = \nu(X) \subset \mathbb{C}^3$. For a generic choice of L the resulting map $X \xrightarrow{\nu} Y$ will be generically $1-1$. Let $\mathcal{C}$ be the conductor of the map ν, and let $\widetilde{\Sigma} \subset \widetilde{Y}$ and $\Sigma \subset Y$ be the locus of the conductor in X, respectively Y. So we have a diagram:

$$\begin{array}{ccccc} \widetilde{\Sigma} & \subset & X & \subset & \mathbb{C}^N \\ \downarrow & & \downarrow \nu & & \downarrow L \\ \Sigma & \subset & Y & \subset & \mathbb{C}^3 \end{array}$$

We put the obvious structure sheaves on $\widetilde{\Sigma}$ and Σ (c.f. (1.5)): $\mathcal{O}_{\widetilde{\Sigma}} = \mathcal{O}_X/\mathcal{C}$ and $\mathcal{O}_\Sigma = \mathcal{O}_Y/\mathcal{C}$. Now, Σ will consist of two parts of different geometric origin:

(1) Σ_1: the image of the double points of the map ν. This will be an ordinary double curve on Y, and hence Σ_1 is reduced.

(2) Σ_2: the image of the curve along which X has points of multiplicity three. Transverse to such a point Y has a $\boldsymbol{D}_4$-singularity, as it is locally the projection of the space curve $Y(3)$ to the plane. A calculation shows that the conductor structure on Σ_2 is also reduced. Note this is no longer the case if we project $Y(m)$, $m \geq 4$, so it is here that the finiteness of T^2_X comes in. (The curve along which X has multiplicity two maps to an ordinary double curve of Y, so there is no conductor coming from this part.)

In [21] and [22] the functor of admissible deformations $\mathrm{Def}\,(\Sigma, Y)$ of the pair $\Sigma \hookrightarrow Y$ was studied. Loosely speaking, $\mathrm{Def}\,(\Sigma, Y)$ consists of deformations of Σ and Y, such that Σ stays inside the singular locus over the deformation. A fundamental result of [22] was that there is a natural equivalence of functors

$$\mathrm{Def}\left(X \longrightarrow \mathbb{C}^3\right) \approx \mathrm{Def}\,(\Sigma, Y).$$

Here $\mathrm{Def}\left(X \longrightarrow \mathbb{C}^3\right)$ is the functor of deformations of the diagram

$$X \longrightarrow \mathbb{C}^3.$$

This functor equivalence follows essentially from

$$\mathrm{Hom}_Y(\mathcal{C}, \mathcal{C}) \approx \mathrm{Hom}_Y(\mathcal{C}, \mathcal{O}_Y) \approx \mathcal{O}_X$$

(see [22]), which means that we can recover the $\mathcal{O}_Y$-module structure and the ring structure of $\mathcal{O}_X$ from the inclusion $\mathcal{C} \hookrightarrow \mathcal{O}_Y$. Note that essentially because $\mathbb{C}^3$ is a smooth space, the forgetful transformation

$$\mathrm{Def}\left(X \longrightarrow \mathbb{C}^3\right) \longrightarrow \mathrm{Def}\,(X)$$

is smooth: there are no obstructions to lifting the three coordinate functions, defining the map to $\mathbb{C}^3$, along with X.

Also, in [21], the notion of I^2-equivalence on the functor $\mathrm{Def}\,(\Sigma, Y)$ was introduced. If Σ is defined in $\mathbb{C}^3$ by an ideal I, and Y is defined by a function f, then two deformations over S described by (I_S, f_S) and (I_S, g_S) are called I^2-equivalent if $f_S - g_S \in I_S^2$. I^2-equivalence is an admissible equivalence relation in the sense of [10], which means that the quotient map

$$\mathrm{Def}\,(\Sigma, Y) \longrightarrow M(\Sigma, Y)$$

is smooth, and the functor $M(\Sigma, Y)$ of I^2-equivalence classes of admissible deformations is semi-homogeneous. Combining these things, we arrive at a diagram

$$\begin{array}{ccc} \mathrm{Def}\left(X \longrightarrow \mathbb{C}^3\right) & \approx & \mathrm{Def}\,(\Sigma, Y) \\ \downarrow & & \downarrow \\ \mathrm{Def}\,(X) & & M(\Sigma, Y) \end{array}$$

The tangent space $M^1(\Sigma, Y) = M(\Sigma, Y)\big(\mathbb{C}[\varepsilon]\big)$ sits in an exact sequence (see [21])

$$0 \longrightarrow I^{(2)}/\big(I^2, \theta_I(f)\big) \longrightarrow M^1(\Sigma, Y) \longrightarrow T^1_\Sigma \longrightarrow \cdots .$$

Here $I^{(2)}$ is the second symbolic power of I, and $\theta_I(f)$ is the ideal generated by $\theta(f)$, where $\theta \in \Theta_I := \{\theta | \theta(I) \subset I\}$. Because Σ is a reduced curve germ, we have

$$\dim\left(T^1_\Sigma\right) \leq \infty$$

and

$$\dim\left(I^{(2)}/I^2\right) \leq \infty,$$

so it follows that $\dim\big(M^1(\Sigma, Y)\big) \leq \infty$. It now follows from Schlessingers theorem that the functor $M(\Sigma, Y)$ has a hull. In other words, $\mathrm{Def}\,(X)$ has a core. ■

Remark 3.6.

(1) We proved the theorem only for surfaces, but clearly something very general is going on. It is natural to expect the theorem to be true for all analytic germs (X,p) such that for some representative X of (X,p) and all $q \in X - \{p\}$ one has that $\mathrm{Def}\big((X,q)\big)$ is smooth. It would be very interesting to prove this in general.

(2) Intuitively it is "clear" that the deformations "high in the series" should give rise to a trivial factor in the base space. One might be tempted to argue along the following lines:

By naturality of the obstruction element of $ob(\xi,\xi') \in T^2$ for $\xi,\xi' \in T^1$ we see that $ob(a.\xi,\xi') = 0$ for all $a \in \mathrm{Ann}\,(T^2)$. So to first order, the subspace $\mathrm{Ann}\,(T^2).T^1 \subset T^1$ is not obstructed against anything. But in general there will be higher order obstructions, or higher order Massey-products (see [29]) non-vanishing, and it is easy enough to give examples where this really happens. Theorem (3.7) states somehow that there is an end to all these Massey-products. It would be interesting to prove the theorem in such a set-up.

The next theorem tells us that this core is really the base space of any series member high in the series.

Stability Theorem 3.7. *Let X be a limit with finite dimensional T^2. Then there is a number n_0 such that for any $n \geq n_0$ and any fibre $X' = X_s$ of a deformation $\mathcal{X} \longrightarrow S \in \mathrm{Sec}^n(S)$ one has:*

$$\mathrm{Core}\,(X) = \mathrm{Core}\,(X').$$

Proof. The idea of the proof is the same as that of (3.5), but for simplicity we assume that X has only transverse $\boldsymbol{A}_1$ singularities. So we again let $X \subset \mathbb{C}^N$ and let $L : \mathbb{C}^N \longrightarrow \mathbb{C}^3$ be a generic linear projection, and we get the diagram

$$\begin{array}{ccccc} \widetilde{\Sigma} & \subset & X & \subset & \mathbb{C}^N \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \subset & Y & \subset & \mathbb{C}^3 \end{array}$$

Let $\mathcal{J} \subset \mathcal{O}_{(\mathbb{C}^N,0)} =: \mathcal{O}_N$ be the ideal of X, so the ideal of Y is

$$\mathcal{J} \cap \mathcal{O}_{(\mathbb{C}^3,0)} = (f) \subset \mathcal{O}_3 := \mathcal{O}_{(\mathbb{C}^3,0)}.$$

Let $l : \mathbb{C}^3 \longrightarrow \mathbb{C}$ be a generic linear function, and let P be the polar curve, that is, the critical locus of the map $(f, l) : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$. So, P is defined by two generic partials of f, say $P = V(\phi)$; $\phi = (\partial f/\partial x, \partial f/\partial y)$. Hence, P is an isolated complete intersection curve singularity. Let $I \subset \mathcal{O}_3$ be the ideal of Σ. Clearly we have the inclusion $\Sigma \subset P$. Let us put $\boldsymbol{m}_k := \boldsymbol{m}_{(\mathbb{C},0)} \subset \mathcal{O}_{\mathbb{C}^k,0})$.

Now, because Σ is reduced, there is an integer p such that

$$I^{(2)} \cap \boldsymbol{m}_3^p \subset I^2.$$

Because P is an isolated complete intersection singularity, it is finitely determined, so we can find an integer q such that

$$\forall \phi' \text{ with } j^q(\phi) = j^q(\phi')\, \exists\, h : \left(\mathbb{C}^3, 0\right) \longrightarrow \left(\mathbb{C}^3, 0\right)$$
$$\text{such that } V(\phi \circ h) = V(\phi').$$

Because the map $X \longrightarrow Y$ is finite, it follows that for all k there is a $n = n(k) \geq k$ such that

$$\left(f, \boldsymbol{m}_3^k\right) \supset \left(\mathcal{J}, \boldsymbol{m}_N^n\right) \cap \mathcal{O}_3.$$

Finally, we let

$$n_0 := n(k); \quad k = \max(p, q+1).$$

Consider a fibre X' of a deformation $\xi \in \mathrm{Sec}^n(X)(S)$. By projection this family $\mathcal{X} \longrightarrow S$ we get families $Y_S \longrightarrow S$, $\Sigma_S \longrightarrow S$ and $P_S \longrightarrow S$. Now because $n \geq n_0$, we have that for each $s \in S$

$$\left(f_s, \boldsymbol{m}_3^k\right) \supset \left(\mathcal{J}_s, \boldsymbol{m}_N^n\right) \cap \mathcal{O}_3 = \left(\mathcal{J}, \boldsymbol{m}_N^n\right) \cap \mathcal{O}_3 \supset \left(f, \boldsymbol{m}_3^n\right).$$

So one has: $f - f_s \in \boldsymbol{m}_3^k \subset \boldsymbol{m}_3^{q+1}$. From this it follows that $j^q(P) = j^q(P_s)$, and hence, P and P_s are isomorphic. We can find a (family of) coordinate transformations, trivializing this family $P_S \longrightarrow S$. Because $\Sigma_S \longrightarrow S$ is a sub-curve (over S) of $P_S \longrightarrow S$, it follows that $\Sigma_S \longrightarrow S$ also can be assumed to be the trivial family. Let I be the ideal of Σ_S in $\mathbb{C}^3 \times S$. We then have

$$f - f_s \in \boldsymbol{m}_3^p \cap I^{(2)} \subset I^2$$

because $p \leq n$. Hence, for each $s \in S$ we have that Y_s is I^2-equivalent to Y. Hence Y and Y_s have the same core. ∎

Remark 3.8. That we really need the family to connect X and X' seems to be a technicality that could be removed with some more care. Also, it is clear from the above argument that one can directly compare the base spaces of fibres X_s and $X_{s'}$ without using the core of the limit as intermediary. Remark also that the result as formulated above is not very practical, because it does not give a hint as to the value of n_0 in terms of X. It would be very useful to have a more effective version of the theorem.

4. Tree Singularities

In this section we will illustrate some of the results of chapter 2 and chapter 3 with the example of the tree singularities. We discuss the geometric content of the generators for T^1 and T^2 that were found in [25]. Furthermore, we describe the simplest class of tree singularities in more detail, to know those which have a simple star as tree. The deformation theory of these singularities leads to the study of configurations of smooth curves in the plane. It offers some insight in the complexity of deformation theory of rational surface singularities. The resulting picture method for understanding the component structure is the subject of a separate paper together with T. de Jong ([26]).

Improvements of Tree Singularities

As in chapter 1, we let $v(T)$, $e(T)$, $o(T)$ and $c(T)$ be the sets of vertices, edges, oriented edges and corners of the tree T. The edges correspond to the irreducible components of the double curve of X, the oriented edges to their inverse images on the normalization. We will use

$$\Sigma_{pq} \subset X_q \quad \text{and} \quad \Sigma_{qp} \subset X_p$$

to denote these curves mapping to $\Sigma_{\{p,q\}}$, $\{p,q\} \in e(T)$ in X. Tree singularities have improvements that are easy to understand: take any embedded resolution of $\cup\Sigma_{qp} \subset X_p$, so we get a diagram

$$\begin{array}{ccc} \Delta_{qp} & \hookrightarrow & Y_p \\ \downarrow & & \downarrow \\ \Sigma_{qp} & \hookrightarrow & X_p \end{array}$$

An improvement is obtained by gluing back:

$$\begin{array}{ccc} \coprod_{(qp)\in o(T)} \Delta_{qp} & \hookrightarrow & \coprod_{p\in v(T)} Y_p \\ \downarrow & & \downarrow \\ \coprod_{\{p,q\}\in e(T)} \Delta_{\{p,q\}} & \hookrightarrow & Y \end{array}$$

The corresponding improvement graph can be characterized by a certain function on the set $o(T)$ of oriented edges

$$\lambda(q,p) = \text{ length of chain from } L_p \text{ to } \Delta_{qp}.$$

We recall here that L_p is the line given by $x = 0$ in the plane X_p, and is transverse the all the other curves.

The series deformations correspond to deforming each of the double curves of the improvement, as explained in chapter 2. When we deform around $\Delta_{\{p,q\}}$, to $\boldsymbol{A}_{\nu(p,q)}$ we get a chain between L_p and L_q of length equal to

$$l(p,q) = \lambda(p,q) + \nu(p,q) + \lambda(q,p).$$

There are two more or less canonical improvements to consider:

$\boldsymbol{M}$: Take for $X_p \longrightarrow Y_p$ the minimal good embedded resolution of $\cup\Sigma_{qp} \subset X_p$. We thus arrive at the minimal good improvement $Y \longrightarrow X$. In this case one has:

$$\lambda(q,p) = \max_r \left(\rho(r,q;p)\right).$$

$\boldsymbol{B}$: Blow-up points of X to arrive at the blow-up model. In this case we have:

$$\lambda(p,q) = \max_{r,s} \left(\rho(r,q;p),\ \rho(s,p;q)\right) = \lambda(q,p).$$

We now come to the relation between the notion of limit tree of [25] of a rational surface singularity with reduced fundamental cycle, and the series of deformations of tree singularities.

Proposition 4.1. *Let $X = X(\boldsymbol{T})$ be a tree singularity of multiplicity m, and let $Y \longrightarrow X$ the $\boldsymbol{B}$-improvement with exceptional divisor E. Let $X' = X_\lambda(\nu)$, $\nu \in \mathcal{N}(Y,E)$ a series member, as in (2.16). Then:*

(1) *X' is a rational surface singularity with reduced fundamental cycle and of multiplicity m.*

(2) *The tree T is a limit tree for X'.*

(3) *The blow-up tree $BT(3)$ of X' is equal to the blow-up tree $BT(3)$ of X.*

Proof. (1) is clear, because X' will be a singularity with hyperplane section the curve $Y(m)$. This condition is equivalent to having reduced fundamental cycle. For (2) we have to recall the definition of a limit tree from [25], Definition (1.12). There T is called a limit tree for X' if the following conditions hold.

(0) The set of vertices of $\boldsymbol{T}$ is equal to the set of $\mathcal{H}$ of irreducible components of the hyperplane section of the singularity.

(1) If $\{p, r\}$ and $\{q, r\}$ are edges of T, then

$$\rho(p, q; r) \leq \rho(q, r; p)$$

$$\rho(p, q; r) \leq \rho(r, p; q)$$

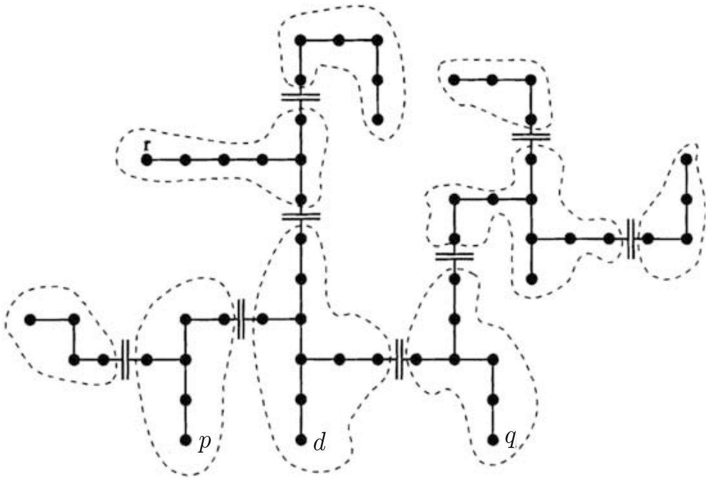
(2) For r and $s \in \boldsymbol{C}_{pq}$, $\{p, r\} \in e(\boldsymbol{T})$ one has:

$$\rho(p, q; r) = \rho(p, s; r).$$

(3) If p, q, r are not on a chain, and if d is the unique center of p, q, r, then one has

$$\rho(p, q; r) \geq \rho(p, q; d).$$

We recall here that the function $\rho(p, q; r)$ is the overlap function of X', that is the number of curves in the minimal resolution of X' that are common to the chains from p to r and q to r. Let us verify these conditions. It is convenient to make a picture; it is more instructive than a formal proof.



Just some improvement graph

(1) One has

$$\rho(p,q;r) \leq \min\big(\lambda(p,r),\ \lambda(q,r)\big)$$
$$\rho(q,r;p) \geq \nu(p,r) + \lambda(r,p)$$
$$\rho(r,p;q) \geq \nu(q,r) + \lambda(q,r)$$

So the inequality is satisfied if $\nu(p,r)$ and $\nu(q,r) \geq \lambda(p,r) - \lambda(r,p)$. But by the symmetry of the λ-function on the $\boldsymbol{B}$-model, this is zero, so it is satisfied for all series members.

(2) This condition is fulfilled for trivial reasons: the function $\rho(p,q;r)$ is determined by the curves that lie in Y_p.

(3) Using (2), we may assume that $\{p,d\}$, $\{q,d\}$ and $\{r,d\}$ are edges of T. In this case one has (see picture):

$$\rho(p,q;r) = \lambda(d,r) + \nu(d,r) + A$$
$$\rho(p,q;d) \leq \lambda(r,d) + A$$

for some A that can be positive or negative. Hence it follows that

$$\rho(p,q;r) - \rho(p,q;d) \geq \lambda(d,r) - \lambda(r,d) + \nu(d,r)$$

which is $\geq \nu(d,r) \geq 0$ by symmetry of λ. Recall that the blow-up tree of a rational singularity is the tree with nodes corresponding to singularities that occur in the resolution process, see [25]. The statement about these blow-up trees is evident: the minimal resolution of the series member is obtained by first deforming the $\boldsymbol{A}_\infty$-singularities to $\boldsymbol{A}_\nu$, and then resolving these. This clearly only changes the blow-up tree by nodes corresponding to singularities of multiplicity 2. ■

Deformations of Tree Singularities

Associated to each edge $\{p,q\}$ of a limit tree of a rational surface singularity with reduced fundamental cycle, there are three elements in T^1 constructed in [25]:

$$\sigma(p,q), \quad \tau(p,q) = \tau(q,p), \quad \sigma(q,p).$$

These $3.(m-1)$ elements generate the T^1 and are subject to m relations, one for each vertex of T:

$$\sum_{p \in \nu(T)} \sigma(p,q) = 0.$$

Let us briefly indicate the method of proof, used in [25]. From the explicit equations and the choice of a limit tree T for X, one constructs elements in the normal module

$$\operatorname{Hom}\bigl(I/I^2, \mathcal{O}_X\bigr).$$

To show that these project onto generators of T^1, one uses the exact sequence relating X with its general hyperplane section, which is the curve $Y(m)$:

$$\ldots T^1_{X/S} \xrightarrow{x.} T^1_{X/S} \longrightarrow T^1_{Y(m)} \longrightarrow \ldots.$$

Here $X \longrightarrow S$ is the slicing of X defined by $x \in \mathcal{O}_X$. So we can test for independence by restriction to $x=0$, and work inside $T^1_{Y(m)}$, which is a very simple space to understand. In this way one can show that the explicitly constructed elements in fact generate.

For tree singularities one of course has analogous elements, and their relations can be described in the same way. It turns out that there is a nice geometrical description for these T^1-generators, and in fact, we first found these geometrical elements for tree singularities, and then found the result for arbitrary rational surface singularities with reduced fundamental cycle by lifting back.

To define the τ-deformations, we take an edge $\{p,q\}$. The corresponding planes X_p and X_q intersect in a smooth curve $\Sigma_{\{p,q\}}$. So $X_p \cup X_q$ is isomorphic to an $\boldsymbol{A}_\infty$-singularity.

Lemma 4.2. *In the coordinates* (1.12)

$$z_{pq}z_{qp} = 0,$$

consider the deformation

$$z_{pq}z_{qp} = f(x)$$

of this $\boldsymbol{A}_\infty$-singularity. If

$$\operatorname{ord}_x(f) \geq s(p,q) := \max_{r,s}\left(\rho(r,p;q),\ \rho(s,q;p)\right),$$

then all the curves $\Sigma_{rp}, \Sigma_{sq}; \{r,p\}, \{s,q\} \in e(T)$ lift over this deformation.

Proof. Let $y = z_{pq}$, $z = z_{qp}$. The $\boldsymbol{A}_\infty$ singularity is described by $yz = 0$ in coordinates x, y, z. A smooth curve in the x, z plane transverse to $x = 0$ can be taken as defined by the ideal $\left(z, y - g(x)\right)$. If we deform the $\boldsymbol{A}_\infty$ and lift the curve, then after coordinate transformation we may in fact suppose that the curve is constant. Hence we must have $(y + \varepsilon.\alpha).z + \varepsilon.\beta.\left(y - g(x)\right) = yz + \varepsilon.f$, hence modulo (y,z) we have that $f \in (g)$. From this the lemma follows. ■

One now can define elements

$$\tau(p,q) \in T^1_X$$

in the following way: deform the $\boldsymbol{A}_\infty$-singularity $X_p \cup X_q$ to $\boldsymbol{A}_{s(p,q)-1}$, where

$$s(p,q) := \max_{r,s}\left(\rho(r,p;q),\ \rho(s,q;p)\right).$$

Lemma 4.2 tells us that we can lift all the curves over this deformation. Now take such a lift, and glue back all the planes to these curves. In this way one gets a deformation of the tree singularity X. By construction, $\operatorname{supp}\left(\mathcal{O}_X.\tau(p,q)\right) = \Sigma_{\{p,q\}}$. For each element $x^n.\tau(p,q)$ one has a one-parameter deformation, that on the level of equations is characterized by:

$$z_{pq}z_{qp} = \lambda.x^{n+s(p,q)}.$$

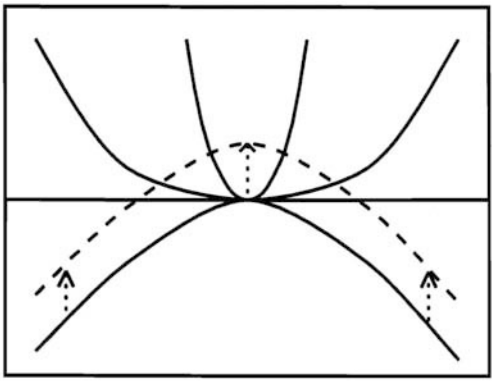
(We note that this description fits with formula (3.10) of [25] for the τ-generators.) These τ-deformations are related to the series deformations in the following way: as mentioned before, deforming on the $(\boldsymbol{B})$-improvement from $\boldsymbol{A}_\infty$ to $\boldsymbol{A}_\nu$ produces a chain of length $\lambda(p,q)+\nu(p,q)+\lambda(p,q) = 2.s(p,q)+\nu(p,q)$. This means that the corresponding T^1-element corresponds to

$$x^{s(p,q)+\nu(p,q)+1}.\tau(p,q).$$

So one sees that the series deformations are contained in the space spanned by the τ's, and form in there a space of finite codimension. Note also that it follows from [25] that the data $(T,\phi,\boldsymbol{f})$ (see (1.13)) describe a rational surface singularity with reduced fundamental cycle exactly if $\mathrm{ord}_x(f_{pq}) \geq s(p,q)$. This gives an alternative way of thinking about the τ-deformations.

Apart from these series deformations, there is for each $(p,q) \in o(T)$ another sort of deformation, that is geometrically even easier to understand than the τ's:

$$\sigma(q,p) \;:\; \text{Move the curve } \Sigma_{qp} \text{ in the plane } X_p.$$



Shifting Σ_{qp} in X_p

These σ's can also be seen as deformations of the decoration:

$$\sigma(q,p) \;:\; a_{qp} \longmapsto a_{qp}+\varepsilon;\;\; \phi(r,q;p) \longmapsto \phi(r,q;p)-\varepsilon.$$

From the interpretation as shiftings of the curves, it now becomes obvious that one has the relations

$$\sum_{p\in\nu(T)} \sigma(p,q) = 0.$$

These just express the fact that shifting in a given plane X_p all the curves Σ_{qp} by the same amount gives a trivial deformation of the tree singularity. It is also clear that the shift deformations are unobstructed among each other.

Obstruction Spaces For Tree Singularities

The generators for T^1 had a natural interpretation in terms of the edges of T. The generators for the obstruction space T^2 have a nice simple description in terms of the ... non-edges of T! to be more precise, there are elements

$$\Omega(p,q) \in T^2_X$$

for each ordered pair $(p,q) \notin o(T)$. It is easy to see that the number of such oriented non-edges is

$$m(m-1) - 2(m-1) = (m-1)(m-2)$$

Furthermore, for each edge $\{p,q\} \in e(T)$ we have a linear relation between the Ω's:

$$\sum_{(p,r;q)\in c(T)} \Omega(p,r) + \sum_{(s,q;p)\in c(T)} \Omega(s,q).$$

By [11], the number of generators of T^2 is $(m-1)(m-3)$, which is indeed the same as $(m-1)(m-2)-(m-1)$.

Let us quickly describe these elements. Recall that T^2 is by definition

$$T^2_X = \mathrm{Hom}\,(\mathcal{R}/\mathcal{R}_0, \mathcal{O}_X)/\mathrm{Hom}\,(\mathcal{F}, \mathcal{O}_X)$$

where

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

is a presentation of $\mathcal{O}_X$ as module over the ambient space. $\mathcal{F}$ is the free module on a set of generators of the ideal of X, $\mathcal{R}$ the module of relations between them, and $\mathcal{R}_0$ the sub-module of the Koszul-relations. In our case $\mathcal{O} := \mathbb{C}\{x, y_1, \ldots, y_m\}$, and equations are provided by (1.12). It is useful to use the notation of the canonical equations, as explained in (1.13). So we write $z_{rp} = z_{r'q}$ in case that $r' \in \boldsymbol{C}_{rq}$, etc. the equations are then simply written as $z_{pq}z_{qp} = 0$, $p \neq q \in v(t)$, and we will use the symbol $[pq]$ to denote the corresponding elements in $\mathcal{F}$. The module of relations is generated by the symbols

$$[p,q;r] := z_{rp}[qr] - z_{rq}[pr] + \phi(p,q;r)[pq].$$

A T^2-element is represented by a homomorphism $\mathcal{R} \longrightarrow \mathcal{O}_X$. Consider the homomorphism

$$\Psi(p,q) := \sum_{a|p\in \boldsymbol{C}_{qa}} z_{qa}[qa]^*.$$

Here $[qa]^* \in \mathcal{F}$ denotes the corresponding element in the dual basis.

A straightforward calculation shows the following:

> $\Psi(p,q)\big([r,s;t]\big) = 0$ unless $t = q$ and the points p, q, r and s lie on a chain in T with r or s between p and q. In that case one has: $\Psi(p,q)\big([r,s;t]\big) = \pm z_{qr}z_{qs}$, ($+$ if $s \in \boldsymbol{C}_{pq}$, $-$ if $r \in \boldsymbol{C}_{pq}$).

But note that if $r \in \boldsymbol{C}_{pq}$, then

$$z_{qr}z_{qs} = \big(z_{sr} + \phi(q,s;r)\big).z_{qs} = \phi(q,s;r).z_{qs}$$

because of the equations. This means that the values of the homomorphism $\Psi(p,q)$ are divisible by some power of x. The power is

$$\rho(p,q) := \min_{r\in \boldsymbol{C}_{pq,r\neq p,q}} \big(\rho(p,q;r)\big),$$

the minimum vanishing order of ϕ-functions of corners "belonging to the chain from p to q". Now choose for each (p,q) an r such that $\rho(p,q;r) = \rho(p,q)$, and define homomorphism

$$\Omega(p,q) := \big[\big(1/\phi(p,q;r)\big)\,\Psi(p,q)\big]$$

($[-]$ = class of in the T^2). Whereas the class of the Ψ's are trivial in the T^2, this is no longer true for the Ω's; in fact they form a system of *generators* for T^2 and this leads to a very beautiful geometrical description of the structure of this module.

The fact that (p,q) is not an edge of T means that the corresponding planes X_p and X_q intersect in a fat point. By an easy explicit computation, one can check that the ideal of this intersection $\mathcal{I}_{pq} = (z_{rs}|r \in C_{sp} \cup C_{sq})$ annihilates the element $\Omega(p,q)$ and so the submodule $\mathcal{O}_X.\Omega(p,q)$ of the T^2 generated $\Omega(p,q)$ is supported on the fat point $X_p \cap X_q$. This is analoguous to the situation with the generators $\sigma(p,q), \tau(p,q)$ of the T^1 that are supported on the intersection curve $X_p \cap X_q$ for edges (p,q) of T.

For a corner $(p,q;r) \in c(T)$, there is only one ϕ on the chain between p and q, and it is easy to see the aforementioned linear relations between then, that arise from an edge.

One can show with the same hyperplane section trick that the Ω's form a generating set for the T^2. But the above explicit description of T^2 annihilating elements will give us an upper bound for the dimension T^2 of the form $\dim(T^2_X) \leq N\big((T,\rho)\big)$, where $N\big((T,\rho)\big)$ is a number that only depends on the discrete data of the limit tree.

In general it is much easier to find lower bounds for the dimension of T^2. for this one has to exhibit the fact that the X under consideration is complicated. One can do this by finding hyperplane sections with high smoothing codimension, or by finding a deformation to a singularity with many singularities. In [25] we proved (theorem 2.13)):

Theorem. *Let X be a rational surface singularity of multiplicity m with reduced fundamental cycle. Let $X_1, X_2, \dots, X_r$ be the singular points of the first blow up $\widehat{X}$ of X. Then there exists a one-parameter deformation over the Artin-component, such that for $s \neq 0$ the fibre X_s has as singularities $X_1, \dots, X_r$, together with one singularity, isomorphic to the cone over the rational normal curve of degree $(-m)$.*

By an application of the semi-continuity of $\dim T^2_X$ one gets

$$\dim T^2_X \geq (m-1)(m-3) + \dim T^2_{\widehat{X}},$$

and so

$$\dim T^2 \geq N(\Gamma),$$

where $N(\Gamma)$ is a number that is easily determined from the resolution graph of X by iteration of the inequality.

In fact, the above theorem is also true for rational surface singularities whose fundamental cycle is reduced except possibly at the (-2) curves. This follows from Laufer's theory of deformations over the Artin-component ([31]), (3.7); one takes as roots the fundamental cycle Z, together with the unions of all curves E_i such that $E_i.Z = 0$, and it is well conceivable that it is true for all rational surface singularities. But rational surfaces with reduced fundamental cycle, and also the tree singularities, have the special property that the above inequalities in fact are *equalities.* Basically this follows from

$$N(\Gamma) = N\big((T,\rho)\big),$$

a purely combinatorial fact. The proof is given in [25], (3.27). In terms of the tree the idea is simple: the $\Omega(p,q)$ live on the fat point, and after each blow-up, the length of this scheme drops by one. The T^2 element lives so long, until a further blow up will separate the planes. In this way the blow-up formula for T^2 looks very natural and obvious. Note that because the blow-up tree $BT(3)$ of X and any series member X' of the $(\boldsymbol{B})$-model are the same, one has stability of T^2.

$$\dim\left(T^2_X\right) = \dim\left(T^2_{X'}\right).$$

Base Spaces And Chain Equations

We now give a description of the "base space" of a tree singularity. These are completely analogous to the equations for the rational surface singularities with reduced fundamental cycle. So let X be a tree singularity, or a rational surface singularity, described by the data $(T, \phi, \boldsymbol{f})$ as described in (1.12), (1.13). We describe Def $(X)(S)$, for any base S as follows.

Let for each $\{p, q\} \in e(T)$ and for each $(p, q; r) \in c(T)$ be given functions

$$F_{pq} \text{ and } \Psi(p, q; r) \in S\{x\} := S \otimes_{\mathbb{C}} \mathbb{C}\{x\}$$

restricting to f_{pq} and $\phi(p, q; r)$ respectively. As we have seen, the F's correspond to series and the Ψ to shift deformations. The F's and Ψ's are heavily obstructed against each other. In fact, a reinterpretation of [25], (4.9) is:

Theorem 4.3. *The system (T, Ψ, F) describes a flat deformation over S if and only if for each oriented chain*

$$p_0, p_1, p_2, \ldots, p_{k-1}, p_k \; ; \{p_i, p_{i+1}\} \in e(T)$$

the following continued fraction "exists as power series in x":

$$\cfrac{F_1}{\Psi_1 + \cfrac{F_2}{\Psi_2 + \cfrac{F_3}{\ddots\, \Psi_{k-1} + \frac{F_k}{\Psi_k}}}}.$$

Here $F_i := F_{p_i, p_{i+1}}$ and $\Psi_i := \Psi_{(p_{i-1}, p_{i+1}; p_i)}$.

Remark 4.4. What happens in practice is that one has some arbitrary deformation Ψ, F given over some power series ring $R := \mathbb{C}\{\boldsymbol{a}\} = \mathbb{C}\{a_1, a_2, \ldots\}$. Each chain gives rise to some ideal in $\mathbb{C}\{\boldsymbol{a}\}$, as follows: every time we have to make a division A/B in the continued fraction, we consider Weierstraß division with remainder:

$$A = \mathcal{Q}.B + \mathcal{R}$$

and then equate to zero the coefficients of the x powers in $\mathcal{R}$. In this way, every chain $\boldsymbol{c}$ defines a unique ideal $J(\boldsymbol{c})$ in R, such that the continued fraction exists over $R/J(\boldsymbol{c})$. Note that $\boldsymbol{c}_1 \subset \boldsymbol{c}_2$ implies that $J(\boldsymbol{c}_1) \subset J(\boldsymbol{c}_2)$. so the ideals are build up inductively, starting from the corners $(p, q; r)$. We have seen that the obstruction space T^2_X was generated by certain elements

$$\Omega(p, q), \quad p, q \notin e(T).$$

Of course, this is no coincidence. It was shown in [25] that the different coefficients of remainders that have to be equated to zero exactly correspond to the elements of T^2. As cyclic quotients have limit trees that are linear chains, and a component structure that is directly related to properties of continued fractions, [13], it is very tempting to try to relate these two types of fractions in some direct way.

The Case Of A Star

Let us analyze further the simplest tree singularities, to know those for which the tree is a star. I.e., there is one central plane X_c, and all other planes intersect X_c in a curve. Example (1.10) is of this type. Let us denote the other planes by X_i, X_j, etc and introduce the short-hand notation $\Sigma_i := \Sigma_{i,c}$, $F_i := F_{i,c}$, $\Psi(i, j) := \Psi(i, j; c)$, etc. So the situation is that we have a bunch of curves Σ_i in the central plane, and planes X_i glued to it. Note that for a star the only non-trivial chains are the corners $(i, j; c)$. So the chain equations become simply:

$$\Psi(i, j) \ \text{divides} \ F_i, \quad \forall\, i, j.$$

These conditions have a very simple geometrical interpretation in terms of the curves Σ_i. As the curve Σ_i is described by the equation $y = a_{ic}$, the x-coordinates of the intersection points of Σ_i and Σ_j are precisely the roots of the function $\Psi(i, j)$. If we consider the function F_i as a function on Σ_i, then this condition just means

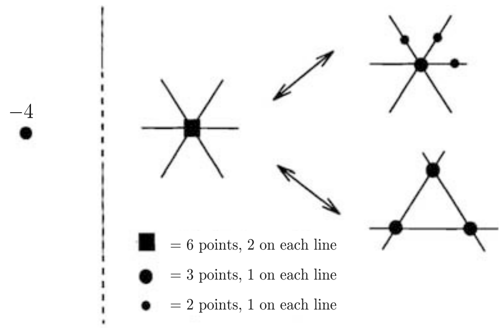
$$(\Sigma_i.\Sigma_j) \subset (F_i), \quad \forall i, j.$$

Here (F_i) denotes the sub-scheme of zero's of the function F_i. Everything now can be understood in terms of these curves and points on these curves. A versal deformation can be described as follows. Let $S := S(F_1)\times\cdots(F_k)\times S(\Sigma)$, where $S(F_i)$ = unfolding space of $F_i \approx \mathbb{C}^{\mathrm{ord}(F_i)}$ and $S(\Sigma) := \delta$-constant stratum in the semi-universal deformation of $\Sigma = \cup_i(\Sigma_i)$. so this is a smooth space. Now look in S for the stratum $\Lambda \subset S$ over which the condition holds.

PICTURES AND COMPONENTS

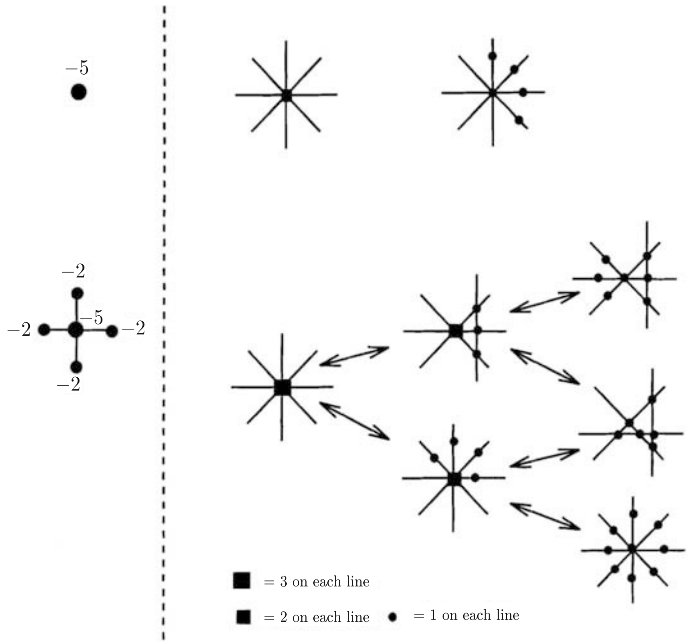
Due to the geometrical nature of the condition one can in the simplest cases understand the component of Λ without any computations. We give some examples.

Example 4.5. We take as a first example the famous Pinkham example, the cone over the (-4), [41]. By (2.19), we can see it as the beginning of a three index series, degenerating into the limit that was described in (1.10).



Pinkham's example

Example 4.6. In a similar way we can see the (-5) as the beginning of a four index series, converging to the tree singularity corresponding to the configuration with four lines in a plane.



The (−5)-series

We define a picture (of $(\Sigma, \boldsymbol{f})$) to be a pair $(\Sigma_s, \boldsymbol{f}_s)$ for some $s \in \Lambda$ such that

(1) Σ_s consists of pairwise transverse intersecting curves.

(2) The zero's of each $f_{i,s}$ are all simple.

It is more or less clear that the combinatorially different pictures correspond to the components of Λ. This also makes it clear that in general there are many components, as long as we take $\mathrm{ord}_x(f_i)$ big enough, that is, high-up in the series: each stratum in the δ-constant deformation of Σ gives a new component. In fact, one can see from the description of Λ that

$$\mathrm{ord}_x(f_i) \geq \sum_{j \neq i} \rho(i,j)$$

is sufficient for base space stability. Here we clearly see that in general T^2 become stable much earlier than the base space itself. For details we refer to the paper [26].

HOMOLOGY OF THE MILNOR FIBRE

There is a nice simple description of the homology of the Milnor fibre of a series member over the smoothing component corresponding to given a picture. To describe this, we need some notation associated to a picture.

Let $\mathcal{P}$ denote the set of distinct points of the picture. So it consists of the points $f_{i,s} = 0$ on the branch Σ_i. Consider the following free $\mathbb{Z}$-modules

$$\boldsymbol{P} := \bigoplus_{p \in \mathcal{P}} \mathbb{Z}.p$$

$$\boldsymbol{L} := \bigoplus_{i} \mathbb{Z}.\Sigma_i$$

There is a natural map

$$\boldsymbol{I} : \boldsymbol{P} \longrightarrow \boldsymbol{L} \quad p \longmapsto \sum_{\{i | p \in \Sigma_i\}} \Sigma_i,$$

mapping each point to the formal sum of the branches containing the point. Let X_s be the Milnor fibre over the component of Λ corresponding to the given picture. Then one has:

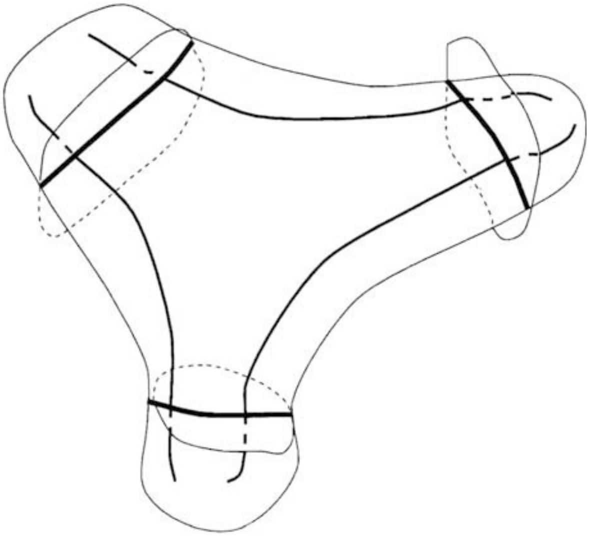
Theorem 4.7. *Let X' be a rational surface singularity with reduced fundamental cycle described by the data $(T, \phi, \boldsymbol{f})$, where the tree T is a simple star. Let M be its Milnor fibre over a smoothing component corresponding to a picture with incidence map $\boldsymbol{I}$. Then one has:*

$$H_1(M) = \mathrm{Coker}\,(\boldsymbol{I}) \quad \textit{and} \quad H_2(X_s) = \mathrm{Ker}\,(\boldsymbol{I}).$$

Proof. (Sketch, for details see [26].) Associated to X' there is a tree singularity X with data (T, ϕ). First we have to take an appropriate small ball, and intersect X with this ball. The space we obtain is topologically a ball D in the central plane, to which we glue some other 4-disc.

We are given a picture as described above. So we can find a one-parameter deformation $X_s \longrightarrow S$ of X, with data $(\Sigma_S, \boldsymbol{f}_S)$, such that for $s \in S - \{0\}$ $(\Sigma_{i,s}, f_{i,s})$ is a picture in the above sense. We can associate to this a two-parameter deformation of X: first use the family Σ_S to shift the curves in the appropriate positions. So this is described by the data $(T, \phi_S, 0)$. Then use $t.f_{is}$ to smooth out the singularities of the spaces X_s. In other words, the two-parameter deformation of X is given by the data $(T, \phi_S, t.\boldsymbol{f}_S)$, $t \in T$, over $S \times T$.

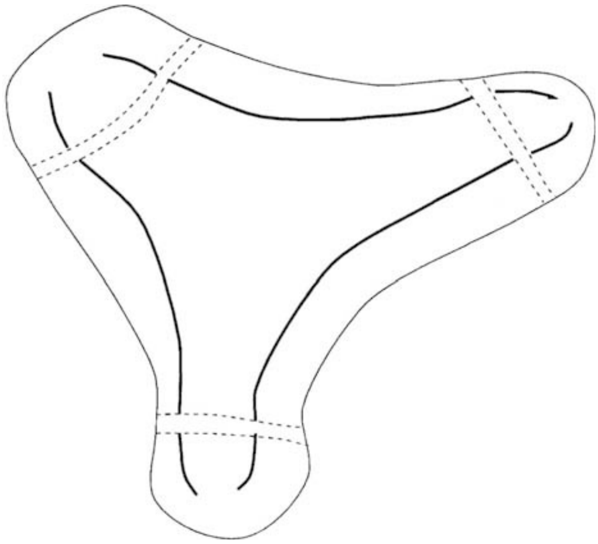
To see clearly what happens it is convenient to blow up D in all the point $\mathcal{P}$. We call this blown-up disc $\widetilde{D}$. Its is homotopy equivalent to a bouquet of spheres. On $\widetilde{D}$ we have the strict transform $\widetilde{\Sigma}$ of the curve Σ. Because all the curves were supposed to intersect transversely, $\widetilde{\Sigma}$ consists of a collection of disjoint curves, each isomorphic to a 2-disc. Now glue the planes back to $\widetilde{D}$. Transverse to each point of $\widetilde{\Sigma}$ this space has an $\boldsymbol{A}_1$-singularity. For the triangle picture of (4.5) it looks something like:



The Milnor fibre X_s is obtained from this space by smoothing out simultaneously these singularities. (This corresponds to deforming $\boldsymbol{A}_\infty$ into $\boldsymbol{A}_{-1}$.) By contracting the discs that were glued in the direction of the central disc, we see that the Milnor fibre is noting but $\widetilde{D}^*$, the space obtained from $\widetilde{D}$ by removing s small tubular neighbourhood $\widetilde{T}$ of $\widetilde{\Sigma}$.

Smoothing out and contracting

For the above example it now looks like:



Cylinders around the curves removed

Now it is easy to compute the homology of the Milnor fibre using the Mayer–Vietoris sequence. We have $\widetilde{D}^* \cap \widetilde{T} = \cup \partial \widetilde{T}_i$, the union of small cylinders around the $\widetilde{\Sigma}$. The sequence now reads:

$$\ldots\ H_2(\widetilde{D}^* \cap \widetilde{T}) \longrightarrow H_2(\widetilde{D}^*) \oplus H_2(\widetilde{T}) \longrightarrow H_2(\widetilde{D}) \longrightarrow H_1(\widetilde{D}^* \cap \widetilde{T})\ \ldots$$

which reduces to

$$0 \longrightarrow H_2(\widetilde{D}^*) \longrightarrow H_2(\widetilde{D}) \longrightarrow H_1(\widetilde{D}^* \cap \widetilde{T}) \longrightarrow H_1(\widetilde{D}^*) \longrightarrow 0.$$

Now, $H_2(\widetilde{D}) = \boldsymbol{P}$ and $H_1(\widetilde{D}^* \cap \widetilde{T}) = \boldsymbol{L}$, where this last isomorphism is set up by mapping the cycle γ_i that runs around Σ_i in the positive direction, to the generator Σ_i of the module $\boldsymbol{L}$. From the geometrical description of the boundary map in the Mayer–Vietoris sequence we get that indeed the resulting map $\boldsymbol{P} \longrightarrow \boldsymbol{L}$ is given by the incidence matrix. ■

Remark 4.8. The picture belonging to the small component of Pinkham's example consists of a triangle, see example 4.5. Hence we get as incidence matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So we see coker $(\boldsymbol{I}) = \mathbb{Z}/2$, ker $(\boldsymbol{I}) = 0$. For the Artin-component one gets as matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

So here coker $(\boldsymbol{I}) = 0$, ker $(\boldsymbol{I}) = \mathbb{Z}$.

Remark 4.9. The onset of stability can be observed very nicely in terms of the points and curves. The set of points $\mathcal{P}$ of a picture decomposes naturally into two pieces: the *imprisoned points* $\cup\Sigma_i \cap \Sigma_j$ and the complementary set of *free points.* If on each curve there is at least one free point, then $H_1 = 0$. From this point on, the Milnor fibre in the series changes only by wedging it with some two-spheres, and so has become stable. The condition for having at least one free point on each branch is that

$$\operatorname{ord}_x(f_i) > \rho(i) := \sum_{j \neq i} \rho(i,j).$$

Note that this point is also exactly the point where the base space itself stabilizes! I do not know whether this relation between base spaces stability and Milnor fibre stability has a more general scope, but it is very well possible. On the other extreme, pictures with the same number of points as curves give rise to smoothings with $\mu = 0$, because a priori we know that the map $\boldsymbol{I}$ must be of maximal rank, as the cokernel must be torsion.

Appendix

We review some basics of deformation theory. We will be very sketchy and this is only meant to be a refresher. For more details we refer to the original literature, like [1], [7], [8], [10], [15], [16], [19], [20], [29], [35], [45], [46], [48]. Let X be any germ of an analytic space.

The Deformation Functor

A deformation of X over a germ $(S, 0)$ is pull-back diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

where $\mathcal{X} \longrightarrow S$ is a flat map. There is an obvious notion of isomorphism between deformations over the same base S.

The deformation functor of X, Def (X), is the functor

$$\mathrm{Def}\,(X)\,:\,\boldsymbol{C}\longrightarrow\mathbf{Set}$$

$$A\longmapsto\big\{\text{Deformations of } X \text{ over Spec}\,(A)\big\}\,/\text{ Isomorphism.}$$

Here $\boldsymbol{C}$ is the category of Artinian $\mathbb{C}$-algebras and **Set** is the category of sets. The category $\boldsymbol{C}$ sits naturally in the category $\boldsymbol{C}_{\mathrm{an}}$ of local analytic $\mathbb{C}$-algebras, which in turn sits in $\widehat{\boldsymbol{C}}$, the formal $\mathbb{C}$-algebras. Our functor in fact is the restriction of a functor defined on these bigger categories, but we will not introduce extra notation to distinguish these functors.

Note the following easy application of the Artin approximation theorem:

Proposition 4.10 (see also [10], (3.1.3.4))**.** *Let X be any germ of an analytic space and let $\widehat{\xi}\in\mathrm{Def}\,(X)\big(\mathbb{C}[[s]]\big)$ a formal deformation. Then for all $n\in\boldsymbol{N}$ there exists a deformation $\xi\in\mathrm{Def}\,(X)(\mathbb{C}\{s\})$ such that the restrictions of $\widehat{\xi}$ and ξ over $\mathbb{C}[[s]]/(s^n)=\mathbb{C}\{s\}/(s^n)$ are the same.*

Proof. Let $X\subset\mathbb{C}^N$, and let $\mathcal{O}=\mathcal{O}_{(\mathbb{C}^N,0)}=\mathbb{C}\{x_1,x_2,\ldots,x_N\}$. Let a presentation of $\mathcal{O}_X$ be given as:

$$\mathcal{O}^\beta\xrightarrow{\ \rho\ }\mathcal{O}^\alpha\xrightarrow{\ \phi\ }\mathcal{O}\longrightarrow\mathcal{O}_X\longrightarrow 0.$$

A formal deformation ξ is given by a $1\times\alpha$ matrix f and an $\alpha\times\beta$ matrix r over the ring $\mathcal{O}[[s]]:=\mathbb{C}[[s]]\otimes_{\mathbb{C}}\mathcal{O}$ that satisfy:

1) $fr=0$ and 2) $\phi(-)=f(s=0,-)$, $\rho(-)=r(s=0,-)$.

Now consider the ring $\mathcal{O}\{s\}[F,R]:=\mathbb{C}\{s,x_1,x_2,\ldots,x_n\}[F,R]$, where $F=(F_1,\ldots,F_\alpha)$, $R=(\ldots,R_{i,j},\ldots)$. In here we have the ideal generated by the components of the matrix $F\cdot R$. The formal deformation ξ gives us a solution $(f,r)\in\mathcal{O}[[s]][F,R]$. By the Artin approximation theorem one now obtains for every $n\in\mathbb{N}$ a solution $(\widetilde{f},\widetilde{r})$ in the ring $\mathcal{O}\{s\}[F,R]$ such that $\widetilde{f}-f=0$ modulo $\boldsymbol{m}^n$ and $\widetilde{r}-r=0$ modulo $\boldsymbol{m}^n$, i.e. we have a convergent deformation of X approximating the given formal one. ■

Functors of Artin Rings

We study functors $F:\boldsymbol{C}\longrightarrow\mathbf{Set}$. Any $R\in ob(\boldsymbol{C})$ gives us a functor $h_R:\boldsymbol{C}\longrightarrow\mathbf{Set}$ via $A\longmapsto\mathrm{Hom}\,(R,A)$. There is a tautological isomorphism

$$F(R)\xrightarrow{\ \approx\ }\mathrm{Hom}\,(h_R,F).$$

In particular, any couple $\mathcal{R} = \big(R, \xi \in F(R)\big)$ gives a map

$$\phi_R \,:\, h_R \longrightarrow F.$$

If there is a couple $\mathcal{R}$ such that $\phi_{\mathcal{R}}$ is an isomorphism, then one says that F is (pro) representable. This is usually a much to strong condition on the functor F.

Definition 4.11. A functor $F \,:\, \boldsymbol{C} \longrightarrow \mathbf{Set}$ is connected iff $F(\mathbb{C}) = \{.\}$. A functor is called semi-homogeneous if the following two "Schlessinger conditions" are satisfied:

If we have a diagram

$$\begin{array}{ccc} & & A'' \\ & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

then the canonical map $F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'')$ is

H.1) surjective if $A'' \longrightarrow A$ is a small surjection;

H.2) bijective if $A = \mathbb{C}$ and $A'' = \mathbb{C}[\varepsilon]$.

Here $\mathbb{C}[\varepsilon] := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ and a small surjection is a map $\alpha \,:\, A'' \longrightarrow A$ such that $\ker(\alpha).\boldsymbol{m_{A''}} = 0$.

For such a functor the tangent space $T^1_F := F\big(\mathbb{C}[\varepsilon]\big)$ acquires in a natural way the structure of a $\mathbb{C}$-vectorspace.

Definition 4.12. A transformation of functors $F \longrightarrow G$ is called smooth if for all small surjections, (hence for all surjections) $B \longrightarrow A$ the induced map

$$F(B) \longrightarrow F(A) \times_{G(A)} G(B)$$

is surjective.

If a transformation ϕ is smooth, and induces an isomorphism $T^1_F \longrightarrow T^1_G$, then ϕ is called minimal smooth. A functor F is called smooth, if the final transformation $F \longrightarrow h_{\mathbb{C}}$ is smooth. A transformation $h_S \longrightarrow h_R$ is smooth if and only if $S = R[[x]]$, the composition of smooth transformations is again smooth, the pull-back of a smooth transformation over an arbitrary transformation is again smooth, etc. So smooth maps are surjections in a very strong sense.

Schlessinger's theorem: If F is a connected semi-homogeneous functor then there exists a minimal smooth transformation:

$$\phi_{\mathcal{R}} : h_R \longrightarrow F$$

if and only if:

H.3) T^1_F is finite dimensional.

Under these circumstances one says that R is a *hull* for F.

Associated to a map $X \xrightarrow{f} Y$, one can consider six deformation functors: Def $(X \xrightarrow{f} Y)$, the deformations of X, Y and f simultaneously, Def (X/Y), deformations of X, f but keeping Y fixed, Def $(X \setminus Y)$, deformations of Y, f, but keeping X fixed, Def (f), deformations only of f, keeping both X and Y fixed. Apart from these one also has Def (X) and Def (Y). There are six cotangent complexes associated to these functors and their homology and cohomology groups sit in various exact sequence, described in detail in the thesis of R. Buchweitz, [10].

In case that X is not a germ, but a global space, there are global $\boldsymbol{T}^i_X$ and local $\mathcal{T}^i_X$ sheaves, related by a usual local-to-global spectral sequence:

$$E_2^{p,q} = H^p(X, \mathcal{T}^q) \Rightarrow \boldsymbol{T}^{p+q}_X.$$

The Base Space of a Limit

We have seen that the base space of a semi-universal deformation of a singularity X appears formally as

$$\mathcal{B}_X = Ob^{-1}(0)$$

for the obstruction map

$$Ob : T^1_X \longrightarrow T^2_X.$$

As for a limit we have $\dim(T^1_X) = \infty$, so the base space of the semi-universal deformation should be infinite dimensional. Working with infinite dimensional spaces causes some inconveniences. There are at least three different attitudes towards these problems possible.

(1) Try to develop a honest analytic theory in infinite dimensions.

In principle, Hausers approach is just achieving this. He uses Banach-analytic methods to construct the base space of a semi-universal deformation for isolated singularities, and his construction works also in the case the T^1 is not finite dimensional. It seems that in the important case that T^2 is finite dimensional, one can use the essentially simpler theory of Mazet [33] of analytic sets of finite definition (i.e. finite number of equations, in infinite number of variables). This would give already quite strong structural statements about the base spaces (finite number of components, etc, see [33]).

(2) Work formally in infinite number of variables.

This is the approach taken in the book of Laudal [29].

(3) Work only with the functor. A functor that satisfies the three Schlessinger conditions has a hull, so "behaves like a finite dimensional space". If we forget about the third Schlessinger condition, we arrive at the notion of a semi-homogeneous functor and these behave more or less as spaces infinite dimension.

We will be very lazy here, and work with the functor approach (3), although a complete development of (1) seems very desirable.

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